MATHEMATICS MAGAZINE

CONTENTS

VOLUME 43 • JANUARY 1970 • NUMBER 1

MM

MATHEMATICS MAGAZINE

STEPHEN A. JENNINGS, *EDITOR*

EDITORIAL CORRESPONDENCE should be sent to the EDITOR, S. A. JENNINGS, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, A. B. WILLCOX, Mathematical Association of America, Suite 310, 1225 Connecticut Avenue, N. W., Washington, D. C. 20036.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Washington D. C., bi-monthly except July–August. Ordinary subscriptions are: 1 year \$4.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of 2 years for \$6.00. Single copies are 80¢.

Second class postage paid at Washington, D. C. and additional mailing offices.

MATHEMATICS MAGAZINE

Published by

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

A JOURNAL
OF COLLEGIATE MATHEMATICS

VOLUME 43
1970

Published at

MENASHA, WIS., AND WASHINGTON, D. C.

POLICY ON BOOK REVIEWS

Readers of MATHEMATICS MAGAZINE will have noticed that very few book reviews have appeared during 1969. The reason for this is that there has been considerable discussion by the publications committee of the MAA on book review policy and the division of responsibility for book reviews between the AMERICAN MATHEMATICAL MONTHLY and this MAGAZINE. At the present time the MONTHLY has a very extensive and satisfactory book review section which provides short notices of most books in mathematics at the collegiate level. Most of the reviews appearing in the MONTHLY are "telegraphic" although more extensive reviews of selected books also appear. To avoid unnecessary duplication MATHEMATICS MAGAZINE will no longer publish very short reviews, and will restrict itself to longer critical reviews of publications of interest to our readers. I would like to take this opportunity of inviting longer critical reviews of textbooks and other publications at the general level of the first two years of college mathematics. I am particularly anxious to have reviews of textbooks based upon classroom experience and I invite any of our readers who feel they might wish to submit such critical reviews for publication to write to me and let me know what they have in mind.

S. A. JENNINGS

TANGENT PLANES AND DIFFERENTIATION

E. L. ROETMAN

It is typical in elementary courses to consider secants and tangents to curves to motivate the definition of the derivative as the limit of the difference quotient. Unfortunately, the student is then usually told that the geometry has nothing to do with it and attention is devoted entirely to the difference quotient. This is entirely understandable since the idea of a tangent is not carefully defined; it is usually defined using the differential coefficient.

This process is wholly inadequate for differentiability in several variables where only continuous differentiability is discussed or a formal definition is given using linear functions without motivation. Tangent planes are often not defined at all but only heuristically "described."

It is our purpose here to give geometric definitions of tangent lines and tangent planes that have intuitive appeal and from them characterize differentiability. We will find that this procedure dictates of necessity the linear "differential" function as the appropriate object to study with respect to differentiation and it clearly illustrates the exact nature of the approximation obtained from the differential. The recent article by Thurston [3] gives a good survey of the problems related to curves and tangents in $\boldsymbol{R}^2$.

POLICY ON BOOK REVIEWS

Readers of MATHEMATICS MAGAZINE will have noticed that very few book reviews have appeared during 1969. The reason for this is that there has been considerable discussion by the publications committee of the MAA on book review policy and the division of responsibility for book reviews between the AMERICAN MATHEMATICAL MONTHLY and this MAGAZINE. At the present time the MONTHLY has a very extensive and satisfactory book review section which provides short notices of most books in mathematics at the collegiate level. Most of the reviews appearing in the MONTHLY are "telegraphic" although more extensive reviews of selected books also appear. To avoid unnecessary duplication MATHEMATICS MAGAZINE will no longer publish very short reviews, and will restrict itself to longer critical reviews of publications of interest to our readers. I would like to take this opportunity of inviting longer critical reviews of textbooks and other publications at the general level of the first two years of college mathematics. I am particularly anxious to have reviews of textbooks based upon classroom experience and I invite any of our readers who feel they might wish to submit such critical reviews for publication to write to me and let me know what they have in mind.

S. A. JENNINGS

TANGENT PLANES AND DIFFERENTIATION

E. L. ROETMAN

It is typical in elementary courses to consider secants and tangents to curves to motivate the definition of the derivative as the limit of the difference quotient. Unfortunately, the student is then usually told that the geometry has nothing to do with it and attention is devoted entirely to the difference quotient. This is entirely understandable since the idea of a tangent is not carefully defined; it is usually defined using the differential coefficient.

This process is wholly inadequate for differentiability in several variables where only continuous differentiability is discussed or a formal definition is given using linear functions without motivation. Tangent planes are often not defined at all but only heuristically "described."

It is our purpose here to give geometric definitions of tangent lines and tangent planes that have intuitive appeal and from them characterize differentiability. We will find that this procedure dictates of necessity the linear "differential" function as the appropriate object to study with respect to differentiation and it clearly illustrates the exact nature of the approximation obtained from the differential. The recent article by Thurston [3] gives a good survey of the problems related to curves and tangents in $\boldsymbol{R}^2$.

Let us start with the simplest case, arithmetic 2-space $\boldsymbol{R}^2$ (the obvious modifications for Euclidean 2-space $\boldsymbol{E}^2$ are left to the reader).

DEFINITION (geometric tangent). *In $\boldsymbol{R}^2$, the line λ through the point P of a pointset $\mathcal{S}$ is tangent to $\mathcal{S}$ at P if P is a limit point of $\mathcal{S}$ and if for every nondegenerate solid cone $\mathcal{K}$ with vertex P and axis Λ there is a ball $\mathcal{B}(P)$ centered at P such that $\mathcal{S}\cap\mathcal{B}(P)\subset\mathcal{K}$.*

In this definition the solid cone means the usual cone-mantel together with the convex set of points containing the axis but within the mantel. The word ball without qualification is an open ball.

DEFINITION (differentiability). *If (1) $\mathcal{S}$ is the graph of a function f whose domain $\mathfrak{D}(f)$ is a subset of the x axis, (2) x_1 is an element of a nondegenerate interval in $\mathfrak{D}(f)$ and (3) $\mathcal{S}$ has a nonvertical tangent at $(x_1, f(x_1))$ then f is said to be differentiable at x_1 and the slope of the tangent line is called the differential coefficient of f at x_1.*

What is the analytic significance of this definition?

THEOREM. *f: $\boldsymbol{R}^1\to\boldsymbol{R}^1$ is a differentiable function at x_1, an interior point of $\mathfrak{D}(f)$, if and only if for every $\epsilon>0$, there is a number m and a $\delta=\delta(\epsilon)$ such that*

$$|f(x)-f(x_1)-m(x-x_1)| < \epsilon|x-x_1|$$

if only $|x-x_1|<\delta$.

(If x_1 is an endpoint of a nondegenerate interval, a corresponding theorem obtains.) This theorem is implied in Thurston [3] and is proved here to prepare later proofs of corresponding theorems. Sufficiency is obtained already in Thurston [2].

Necessity. Suppose the tangent line Λ has slope m and equation $y=y_1+m(x-x_1)$, $y_1=f(x_1)$. Then the cone-mantel with axis Λ is given by $y_+=y_1+(m+\alpha)(x-x_1)$, $y_-=y_1+(m-\alpha)(x-x_1)$.

By definition, for every α, there is a ball $\mathcal{B}(x_1;\alpha)=\{(x,y)\mid(x-x_1)^2+(y-y_1)^2<r_\alpha^2\}$ such that $(x,f(x))\in\mathcal{B}(x_1;\alpha)$ implies $(x,f(x))$ is in the cone determined by α. Thus for every $\alpha>0$, there is a $\delta>0$, $(\delta=r_\alpha(1+a^2)^{-1/2}$, $a=\max(|m+\alpha|, |m-\alpha|))$ such that for $0\leqq x-x_1<\delta$,

$$(m-\alpha)(x-x_1)+y_1<f(x)<(m+\alpha)(x+x_1)+y_1$$

and for $0\leqq x_1-x<\delta$,

$$(m+\alpha)(x-x_1)+y_1<f(x)<(m-\alpha)(x-x_1)+y_1.$$

(The δ is determined by projecting the points of the intersection of the circle $(x-x_1)^2+(y-y_1)^2=r_\alpha^2$ and the cone-mantel onto the x axis.) This gives,

$$-\alpha|x-x_1| < f(x)-y_1-m(x-x_1)<\alpha|x-x_1|,$$

or

$$|f(x)-f(x_1)-m(x-x_1)| < \alpha|x-x_1|.$$

Thus the notion of tangency is not just a motivation for differentiation but differentiation is an outgrowth of an adequate definition of tangency. A figure will also illustrate clearly what is meant by the local linear approximation properties of differentiation.

For consistency with our later discussion we define the derivative of f at x_1 to be the homogeneous linear part of the function defining Λ, i.e., $Df(x_1, h) = f'(x_1)h$.

Dimension >2. The elementary cones considered in the previous paragraphs are not quite the objects that we find most useful here. Let us first redefine the word cone. By an open semicone with vertex at a point $\bar{x}_0$, $\mathcal{K}(\bar{x}_0)$, we mean a set of points such that $\bar{x}_0+\bar{s}\in\mathcal{K}(\bar{x}_0)$, $\bar{x}_0+\bar{t}\in\mathcal{K}(\bar{x}_0)$ implies that $\bar{x}_0+(\bar{s}+\bar{t})\in\mathcal{K}(\bar{x}_0)$ and $\bar{x}_0+\alpha\bar{t}\in\mathcal{K}(\bar{x}_0)$ for $\alpha>0$. Notice that the interior of each nappe of an elementary cone is an open semicone. Another example of an open semicone is an open half space whose bounding hyperplane contains x_0 as well as finite intersections of such hyperplanes which we will call polyhedral open semicones. If we define open cones as the union of an open semicone and its reflection through $\bar{x}_0$, then the discussion of the above paragraphs could be made using the closures of these open cones.

Having extended our concept of cones we define a *cocone* with vertex at $\bar{x}_0$, $\mathcal{C}(\bar{x}_0)$, as the complement of a cone at $\bar{x}_0$. A cocone will be called nondegenerate if it contains the complement of an open elementary $\boldsymbol{R}^p$-cone with vertex angle $<\pi/2$.

We observe that in $\boldsymbol{R}^2$ a closed elementary cone is a cocone by our present formulation; it is even a polyhedral cocone. In $\boldsymbol{R}^3$ the set of points (x, y, z) such that

$$z = z_0 + m_1(x - x_0) + m_2(y - y_0) \pm \alpha[(x - x_0)^2 + (y - y_0)^2]^{1/2}, \qquad \alpha > 0$$

represents the mantel of a cone with vertex (x_0, y_0, z_0). The corresponding cocone $\mathcal{C}(\bar{x}_0, \alpha)$ contains the punctured plane Π where the equation for Π is $z-z_0 = m_1(x-x_0)+m_2(y-y_0)$ for all $\alpha>0$. Other examples of simple cocones containing Π have mantels given by

$$z = z_0 + m_1(x - x_0) + m_2(y - y_0) \pm \beta_1|x - x_0| \pm \beta_2|y - y_0| \qquad (\beta_1, \beta_2 > 0),$$

polyhedral cocones.

DEFINITION (geometrical tangent plane). *In $\boldsymbol{R}^p$, the plane Π through the point $\bar{x}_0$ of a point set $\mathcal{S}$ is tangent to $\mathcal{S}$ at $\bar{x}_0$ if $\bar{x}_0$ is a limit point of $\mathcal{S}$ and if for every nondegenerate cocone $\mathcal{C}(\bar{x}_0)$ with vertex $\bar{x}_0$ containing Π in its interior, there is a ball $\mathcal{B}(\bar{x}_0)$ centered at $\bar{x}_0$ such that $\mathcal{S}\cap\mathcal{B}(\bar{x}_0)\subset\mathcal{C}(\bar{x}_0)$.*

The reader will find it instructive to prove that the tangent plane to a sphere or ellipse as usually defined satisfies this definition.

DEFINITON (differentiability). *A function f: $\boldsymbol{R}^p\to\boldsymbol{R}^1$ is said to be differentiable at $\bar{x}_0\in\boldsymbol{R}^p$ if (1) $\bar{x}_0$ is an element of the domain of f, $\mathfrak{D}(f)$, and an accumulation point of an open set in $\mathfrak{D}(f)$, (2) the graph of f, $\{(\bar{x}, z=f(\bar{x}))\}$ in $\boldsymbol{R}^p\times\boldsymbol{R}^1=\boldsymbol{R}^{p+1}$ has a tangent plane at $\bar{x}_0$ not parallel to the z axis.*

THEOREM (equivalence). *A function f is differentiable at $\bar{x}_0$ where $\bar{x}_0$ is interior to $\mathfrak{D}(f)$ if and only if for every $\epsilon>0$, there is a $\delta>0$ and real numbers $m_i(i=1, \cdots, p)$ such that*

$$\left|f(\bar{x}) - f(\bar{x}_0) - \sum_{i=1}^{p} m_i(x_i - x_{0i})\right| < \epsilon\|\bar{x} - \bar{x}_0\|_p$$

if only $\bar{x}\in\mathfrak{D}(f)$ and $\|\bar{x}-\bar{x}_0\|<\delta$.

We define the m_i to be the *differential coefficients* of f at $\bar{x}_0$ and the *derivative* to be the homogeneous linear function $Df(\bar{x}_0, \bar{h}) = \sum m_ih_i$.

Suppose now that $\mathcal{S}$ is the graph of a function $z=f(\bar{x})$ defined in a neighborhood of $(\bar{x}_0)$. Suppose $\mathcal{S}$ has a tangent plane Π at $(\bar{x}_0, z_0=f(\bar{x}_0))$ with equation

$$z = z_0 + \sum_{i=1}^{p} m_i(x_i - x_{0i})$$

and cocone mantel

$$z = z_0 + \Sigma m_i(x_i - x_{0i}) \pm \alpha\|\bar{x} - \bar{x}_0\|_p.$$

The definition implies that for every $\alpha>0$, there is an $r_\alpha>0$, such that $(z-z_0)^2+\|\bar{x}-\bar{x}_0\|_p^2<r_\alpha^2$ implies that

$$-\alpha\|\bar{x} - \bar{x}_0\|_p \leqq f(\bar{x}) - z_0 - \sum_{i=1}^{p} m_i(x_i - x_{0i}) \leqq \alpha\|\bar{x} - \bar{x}_0\|_p.$$

Clearly for $(\bar{x}, z)$ in the cocone, $|z-z_0| \leqq (M+\alpha)\|\bar{x}-\bar{x}_0\|_p$ where $M=\max(|m_i|, i=1, \cdots, p)$. Thus for

$$\|\bar{x} - \bar{x}_0\|_p \leqq \delta(1 + (M + \alpha)^2)^{-1/2},$$

$$\left|f(\bar{x}) - z_0 - \sum_{i=1}^{p} m_i(x_i - x_{0i})\right| \leqq \alpha\|\bar{x} - \bar{x}_0\|_p.$$

The sufficiency is easily established. One identifies m_i as $\partial_{x_i}f$ by considering components in the usual manner.

This further illustrates that elementary considerations of geometry lead immediately to the concepts of differentiability most consistent with our modern viewpoint as expressed, say, in Nevanlinna [**1**].

Vector functions. Since our interest is in the geometric foundation for differentiation, we shall not go into the general problem of geometric tangent planes of lower order but only those related to graphs. To set the stage we consider the tangent to a parametrized curve in $\boldsymbol{R}^2$. The geometric tangent to the $\boldsymbol{R}^2$ trace of a curve at a point has been defined above. To translate this into analytic criteria while focusing attention on the trace introduces all of the difficulties of the representation problem, Thurston [**2**], [**3**], Ward [**4**]. In fact, the geometric tangent is not adequate to give a definition of differentiability, Thurston [**2**]. One way to proceed is to introduce kinematical considerations, Thurston [**3**]. We proceed by a very different route. We are not interested only in the tangent to the trace of the curve but in the differentiability of the parametric representation. To accomplish this we consider the graph of a representation function in $\boldsymbol{R}^3 = \boldsymbol{R}^1 \times \boldsymbol{R}^2$.

Consider the parametrization of a curve C:

$$(\phi\colon t \in I \subset \boldsymbol{R}^1 = T) \to (\bar{x} \in \boldsymbol{R}^2 = X^2),$$

$\bar{x} = (\phi_1(t), \phi_2(t))$ where $I = (0, 1)$. Let $\|x\|_2 = \max(|x_1|, |x_2|)$ be the norm in X^2 and $|t| + \|x\|_2$ the norm in $T \times X^2$.

DEFINITION (tangent to the graph). *Consider the graph of ϕ, $\mathcal{G}(\phi)$, in $T \times X^2$ at $(t_0, \bar{x}_0 = \phi(t))$, $t_0 \in I$. Suppose there is a pair of linearly independent 2-planes Π_1 and Π_2 in $T \times X^2$ passing through $(t_0, \bar{x}_0)$ such that for every pair of nondegenerate cocones $\mathcal{C}_1(t_0, \bar{x}_0)$ and $\mathcal{C}_2(t_0, \bar{x}_0)$, vertex $(t_0, \bar{x}_0)$, containing Π_1 and Π_2 respectively, there is a ball $\mathcal{B}(t_0, \bar{x}_0)$ with the property that $\mathcal{G}(\phi) \cap \mathcal{B}(t_0, \bar{x}_0) \subset \mathcal{C}_1(t_0, \bar{x}_0) \cap \mathcal{C}_2(t_0, \bar{x}_0)$. The line of intersection Λ of Π_1 and Π_2 is called the tangent line to $\mathcal{G}(\phi)$ at $(t_0, \bar{x}_0)$.*

We now want to define what we mean by differentiability. The condition that corresponds to the nonvertical tangents in the previous definitions is that Λ not be parallel to the X^2-plane. Certainly for a *regular point* we want the projection of Λ onto the X^2 plane to be a tangent to the trace at $\bar{x}_0$. This means that Λ should not be parallel to the T axis. Thus

DEFINITION (differentiability). *The function ϕ: $I \to X_2$ is differentiable at $t_0 \in I$ if the graph $\mathcal{G}(\phi)$ has a tangent line Λ at $(t_0, \bar{x}_0 = \phi(t_0))$ which is not parallel to the X^2-plane. ϕ is called regular if Λ is not parallel to the T-axis.*

THEOREM. *The representation ϕ of the curve C is differentiable at $t_0 \in I$ if and only if a vector $\bar{\mu} = (\mu_1, \mu_2)$ exists such that for every $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ for which*

$$\|\bar{x} - \bar{x}_0 - \bar{\mu}(t - t_0)\|_2 < \epsilon |t - t_0|$$

if only $|t - t_0| < \delta$.

We discuss only two of the difficult parts of the necessity: the existence of δ from the existence of the ball and obtaining $\bar{\mu}$. Let the mantels of $\mathcal{C}_1$ and $\mathcal{C}_2$ be given by

$$|m_{11}(x_1 - x_{01}) + m_{12}(x_2 - x_{02}) - n_1(t - t_0)| = \alpha_1(|t - t_0| + \|\bar{x} - \bar{x}_0\|_2)$$

and

$$|m_{21}(x_1 - x_{01}) + m_{22}(x_2 - x_{02}) - n_2(t - t_0)| = \alpha_2(|t - t_0| + \|\bar{x} - \bar{x}_0\|_2)$$

respectively. Using

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \qquad \overline{N} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \qquad \overline{A} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

we have

$$* \, [\overline{N} \mid -M]\begin{bmatrix} t - t_0 \\ \bar{x} - \bar{x}_0 \end{bmatrix} = \pm \, \overline{A}(\mid t - t_0 \mid + \|x - x_0\|_2).$$

The condition that Λ be not parallel to X^2 means that M^{-1} exists and

$$\|x - x_0\|_2 \leqq \|M^{-1}\overline{N}\| \mid t - t_0 \mid + \|M^{-1}\overline{A}\|(\mid t - t_0 \mid + \|x - x_0\|_2).$$

That is,

$$(1 - \|M^{-1}\overline{A}\|)\|x - x_0\|_2 \leqq (\|M^{-1}\overline{N}\| + \|M^{-1}A\|) \mid t - t_0 \mid .$$

If α_1, α_2 are small enough, $1 - \|M^{-1}\overline{A}\| > 0$ and

$$\|x - x_0\|_2 \leqq (\|M^{-1}\overline{N}\| + \|M^{-1}\overline{A}\|)(1 - \|M^{-1}\overline{A}\|)^{-1} \mid t - t_0 \mid .$$

Thus

$$\mid t - t_0 \mid < r_\alpha(1 - \|M^{-1}\overline{A}\|)(1 + M^{-1}\overline{N})^{-1}$$

and $(t - t_0, x - x_0) \in \mathcal{C}_1 \cap \mathcal{C}_2$ implies that $\|x - x_0\|_2 + \mid t - t_0 \mid < r_\alpha$.

From $*$ above, the cocone conditions give

$$\begin{aligned} \|\bar{x} - \bar{x}_0 - M^{-1}\overline{N}(t - t_0)\|_2 &< \|M^{-1}\overline{A}\|(\mid t - t_0 \mid + \|x - x_0\|_2) \\ &< \|M^{-1}\overline{A}\|(1 + \|M^{-1}\overline{N}\|)(1 - \|M^{-1}\overline{A}\|)^{-1} \mid t - t_0 \mid . \end{aligned}$$

Given $\epsilon > 0$, choose $\overline{A} = (\alpha_1, \alpha_2)$ so that $\|M^{-1}\overline{A}\|(1 + \|M^{-1}\overline{N}\|)(1 - \|M^{-1}\overline{A}\|)^{-1} < \epsilon$. Let r_α be the radius of the ball $\mathcal{B}(t_0, \bar{x}_0)$ corresponding to this choice of $\overline{A}$ and then choose

$$\delta = r_\alpha(1 - \|M^{-1}\overline{A}\|)(1 + M^{-1}\overline{N})^{-1}.$$

Define $\bar{\mu} = M^{-1}\overline{N}$. The condition that Λ be not parallel to T means that $\overline{N} \neq 0$ and hence that $\mid \mu_1 \mid + \mid \mu_2 \mid > 0$. (Compare this natural result of the geometry with the explanation of this condition in standard advanced calculus books.)

A sketch will convince the reader that the condition for a tangent line Λ is equivalent to the existence of a nondegenerate cone with vertex at (t_0, x_0) and Λ as a central axis. We use the intersecting planes in order to help the reader to the final definition below. Let $(f\colon A \subset \boldsymbol{R}^m = X) \rightarrow (\boldsymbol{R}^p = Y)$ be defined on a set A with interior. Let $\mathcal{G}(f) = \{(x, y) \mid x \in A, y \in Y, y = f(x)\}$ be the graph of f in $X \times Y$. Let $\|x\|_m$ be the max norm in X and $\|y\|_p$ the max norm in Y and $\|x\|_m + \|y\|_p$ the norm in $X \times Y$.

DEFINITION (tangent plane). *The graph $\mathcal{G}(f)$ has a tangent plane Π at a point $(x_0, y_0 = f(x_0))$ interior to $\mathcal{G}(f)$ if there are p $(m+p-1)$-planes Π_i, $i = 1, \cdots, p$ such that $\Pi = \bigcap_{i=1}^{p} \Pi_i$ and for arbitrary nondegenerate cocones $\mathcal{C}_i((x_0, y_0))$, $i = 1, \cdots, p$ associated with the planes Π_i there is a ball $\mathcal{B}(x_0, y_0)$ for which*

$$\mathcal{G}(f) \cap \mathcal{B}(x_0, y_0) \subset \bigcap_{i=1}^{p} \mathcal{C}_i(x_0, y_0).$$

DEFINITION (differentiation). *We say that f is differentiable at x_0 if there is a tangent plane Π to $\mathcal{G}(f)$ at x_0 which is not parallel to Y.*

THEOREM. *f is differentiable at x_0 if and only if there is a $p\times m$ matrix M such that for every $\epsilon>0$, there is a $\delta=\delta(\epsilon)>0$ with*

$$\|f(x)-f(x_0)-M(x-x_0)\|_p<\epsilon\|x-x_0\|_m$$

if only $\|x-x_0\|_m>\delta$.

Consider the cocone mantels associated with Π_i, $i=1,\cdots,p$,

$$-\sum_{\mu=1}^{m} n_{i\mu}(x_\mu-x_{0\mu})+\sum_{k=1}^{p}\tilde{m}_{ik}(y_k-y_{0k})=\pm\,\alpha_i(\|x-x_0\|_m+\|y-y_0\|_p).$$

Let N be the $p\times m$ matrix $[n_{i\mu}]$, $\tilde{M}$ the $p\times p$ matrix $[\tilde{m}_{ik}]$ and A the vector $[\alpha_i]$. The condition that Π is not parallel to Y implies that $\tilde{M}^{-1}$ exists. Thus

$$\|y-y_0\|_p\leqq(\|\tilde{M}^{-1}A\|+\|\tilde{M}^{-1}N\|)\|x-x_0\|_m+\|M^{-1}A\|\|y-y_0\|_p$$

and, for the α_i small enough,

$$\|y-y_0\|_p\leqq(\|\tilde{M}^{-1}A\|+\|\tilde{M}^{-1}N\|)(1-\|\tilde{M}^{-1}\overline{A}\|)^{-1}\|x-x_0\|_m.$$

Finally then, the condition of differentiability means that

$$\|f(x)-y_0-\tilde{M}^{-1}N(x-x_0)\|_p<(1+\|\tilde{M}^{-1}N\|)(1-\|\tilde{M}^{-1}A\|)^{-1}\|x-x_0\|_m$$

if $\|x-x_0\|\leqq(1-\|\tilde{M}^{-1}A\|)(1+\|\tilde{M}^{-1}N\|)^{-1}r$, r being the radius of the ball $\mathcal{B}(x_0,y_0)$. Set $M=\tilde{M}^{-1}N$. The converse is again straightforward.

We call f regular if $m\leqq p$ and M has rank m, that is N has rank m. This is again just the analytic expression of the geometric condition that the projection of Π into Y contains an m dimensional plane in Y.

Examining components it is easy to see that $m_{i\mu}=\partial_\mu f_i$. In fact, one can choose $\Pi_1,\cdots,\Pi_p$ such that $\tilde{M}$ is the identity matrix and $N=M$.

The reader will observe that the dimension of the spaces does not play an essential role in most cases of the above discussion, whence the extension to Banach spaces is immediate. We will also find it interesting to reconsider the Dini coefficients, directional derivatives and their generalization from this geometric viewpoint.

References

1. F. and R. Nevanlinna, Absolute Analysis, Springer, Berlin, 1959.

2. H. A. Thurston, On the definition of a tangent-line, Amer. Math. Monthly, 71 (1964) 1099–1103.

3. ———, Tangents: an elementary survey, this MAGAZINE, 42 (1969) 1–11.

4. A. J. Ward, On Jordan curves possessing a tangent everywhere, Fund. Math., 28 (1937) 280–288.

BINARY RELATIONS AS BOOLEAN MATRICES

OSKAR FEICHTINGER and B. L. McALLISTER, Montana State University, Bozeman

That binary relations can be interpreted as matrices was pointed out at about the turn of the century by Ernst Schröder [1]. This interpretation is not difficult, but, though it is known to experts, it seems to have been forgotten among college teachers in general. We shall give an elementary exposition of this point of view and some of its simpler consequences.

1. A binary relation is, roughly, a statement involving two variables. When the variables are replaced by names of things, the statement may become true or it may become false, depending on what things are chosen. For mathematical work it is difficult and *unnecessary* to insist on working with the meaning of the relation in the intentional sense, and we may do all our work with the extensional meaning. What this means is that we may regard a relation as being completely determined by its extension, i.e., by the set of things which make the statement true. From this point of view, nothing is lost if we take the usual next step of identifying the relation with its extension. Since a binary relation is a statement in two variables, its extension is a class of ordered pairs. Accordingly, a *binary relation* is defined to be any set whose elements are ordered pairs. The classical examples of binary relations in elementary mathematics are order relations, equivalence relations, and functions. Of course when we say that these "are" binary relations we are still taking the set-theoretic or extensional point of view, so that, for example, a (single-valued) function is a set f of ordered pairs such that if $(a, b)\in f$ and $(a, c)\in f$ then $b=c$. In general we shall call a binary relation R *single-valued* if it has this property. Thus we identify single-valued functions with single-valued binary relations. As a matter of convenience, in the rest of this paper, the word "function" will mean single-valued function. Further, since the only relations we shall be explicitly concerned with are binary relations, we shall let the word *binary* be understood. A *function* is then describable as a single-valued relation.

2. If R is a relation we call the set of all first elements of ordered pairs in R the *domain* of R, and we say that R is *on* its domain. This terminology is standard for functions. We shall also say that R is *in* any set that contains the domain of R. This use of *in* is not standard, but is strongly suggested by the standard use of *into* and *onto*.

We shall denote by $\hat{R}$ the *reverse* (also called the *inverse*) of binary relation R. I.e., $\hat{R}$ is the set of all (a, b) such that $(b, a)\in R$. The domain of $\hat{R}$ is called the *range* of R, and we say that R is *onto* its range and *into* any set that contains the range of R. Recalling that the Cartesian product $A\times B$ is just the set of all ordered pairs (a, b) with $a\in A$ and $b\in B$, one sees that a relation R is in A into B if and only if R is a subset of $A\times B$.

3. If A is a set and B is a subset of A, the characteristic function of B is that function f on A into the set $\{0, 1\}$ for which $f(x)=1$ if $x\in B$ and $f(x)=0$ if $x\in A-B$. Here it will be convenient to think of 0 and 1 as comprising a Boolean algebra, so that $0\cdot 0=0\cdot 1=1\cdot 0=0+0=0$ as usual, $0+1=1+0=1\cdot 1=1$ as usu-

al, and $1+1=1$. (We are using $\cdot$ and $+$ to represent the operations now usually denoted by $\wedge$ and $\vee$, respectively. This will make some later formulas seem more familiar.) One reason we wish to interpret 0 and 1 as forming a Boolean algebra is that then the characteristic function of the union of two subsets is the sum of the characteristic functions of the subsets, and the characteristic function of the intersection of two subsets is the product of the characteristic functions of the subsets.

For a fixed set A, each subset is obviously completely determined by its characteristic function. Thus, in particular, a relation R in a set A into a set B is uniquely determined by its characteristic function as a subset of $A \times B$, i.e., by the function f on $A \times B$ into $\{0, 1\}$ which assigns the value 1 to each ordered pair (a, b) in R and the value 0 to each ordered pair (a, b) in $(A \times B) - R$.

Now a matrix may also be viewed as a function defined on a Cartesian product. For example, if M is an $m \times n$ matrix with real elements, we may let $A = \{1, 2, \cdots, m\}$, $B = \{1, 2, \cdots, n\}$, and think of M as a function f on $A \times B$ into the set of real numbers, interpreting $f(i, j)$ as that element of M that lies in the ith row and jth column. From this point of view we see that the characteristic function of a relation R in A into B is essentially a matrix with elements in the Boolean algebra $\{0, 1\}$. For example, the relation "is less than" in the positive integers into the positive integers has as characteristic function the matrix

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & \cdots \\
\hline
1 & 0 & 1 & 1 & 1 & 1 & \cdots \\
2 & 0 & 0 & 1 & 1 & 1 & \cdots \\
3 & 0 & 0 & 0 & 1 & 1 & \cdots \\
4 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & & & \cdots & & &
\end{array}
$$

("Garden variety" matrices are usually $m \times n$ matrices with both m and n finite. However, as long as both rows and columns are indexed in some clear-cut way we needn't bother about this finiteness, especially since most of our matrices are infinite. On the other hand, this indexing is important, which is why we displayed the indices explicitly.)

4. If R is a relation in A into B, let us denote the characteristic function (matrix) of R by R_M. Then R_M is a function on $A \times B$ into $\{0, 1\}$ and we may justifiably use the notation $R_M = R_M(i, j)$ where i and j are "independent variables" ranging, respectively, through A and B. If we fix $i_0 \in A$, we may think of $R_M(i_0, j)$ as specifying a function f on B into $\{0, 1\}$. That is, as i_0 remains fixed and j ranges through B, the formula $R_M(i_0, j)$ determines f on B into $\{0, 1\}$. This f may be regarded as the i_0th row of R_M, since f and the i_0th row of R_M determine each other uniquely. To indicate the dependence of f on R_M and

i_0, we shall also write $f = R_M(i_0,\ )$. In a similar fashion we may define for fixed $j_0 \in B$ a function $R_M(\ , j_0)$ on A into $\{0, 1\}$, and this function will be called the j_0th column of R_M.

Now the rows and columns of R_M have a simple interpretation in terms of the relation R itself. To see this, note that the row $R_M(i_0,\ )$ is a function on B into $\{0, 1\}$, hence is the characteristic function of some subset of B. But what subset? Well, exactly the set of all $j \in B$ for which $R_M(i_0, j) = 1$. But this is precisely the set of all those $j \in B$ for which $(i_0, j) \in R$. This latter set is often called the *image* of i_0 under R, and denoted by $R[i_0]$. In other words, the i_0th row of R_M is the characteristic function of the image $R[i_0]$ of i_0 under R, regarding $R[i_0]$, of course, as a subset of B. (If R is a *function* on A, there is, technically, a subtle difference between $R[i_0]$ as we use it and $R(i_0)$ as it is usually interpreted. Namely in this case, our $R[i_0]$ will be a subset C of B and C will have exactly one element. In the usual interpretation, $R(i_0)$ denotes that one element, the unique member of C. In some contexts this distinction is worth fussing about, but not in ours. However, it is why we replace the parentheses of functional notation by brackets.)

In the same way the j_0th column of R_M is the characteristic function of the *preimage* $\hat{R}[j_0]$ of j_0, i.e., of the set of all $i \in A$ such that $(i, j_0) \in R$.

With rows and columns of R_M defined, we may speak of the transpose $R_M{}^T$ of R_M obtained by exchanging the rows with the columns of R_M. More precisely, $R_M{}^T(i, j) = R_M(j, i)$. But it is then easy to see that the transpose of R_M is the matrix of the reverse relation $\hat{R}$ of R. That is, $(\hat{R})_M = (R_M)^T$.

In case $A = B$ so that rows and columns are indexed by the same set, it makes sense to speak of *symmetry* of a matrix. Namely, a matrix is called *symmetric* if it is equal to its transpose, and thus R_M is symmetric if and only if it is equal to $(\hat{R})_M$. But this happens if and only if $R = \hat{R}$. Now, a relation R is called *symmetric* if $(a, b) \in R$ entails $(b, a) \in R$; but this is equivalent to the condition $R = \hat{R}$. Thus R is symmetric as a relation if and only if R_M is symmetric as a matrix.

Another condition for relations in A into B that is of interest when $A = B$ is that of reflexivity. A relation R in A into A is *reflexive* provided that for each $a \in A$ it is true that $(a, a) \in R$. But this is precisely the condition that R_M "have 1's all down the diagonal," i.e., that for each $a \in A$, $R_M(a, a) = 1$. Antireflexivity of R is similarly equivalent to the property of R_M of having all 0's down the diagonal. Antisymmetry is similarly easy to characterize. The property of R of being transitive is interpretable in terms of the positions of 1's in the matrix R_M, but the interpretation is not simple enough to justify our pursuing it here.

5. If X is a subset of A we may think of the values of the characteristic function κ of X at points of A as analogues of the components of a vector. That is, let us speak of $\kappa(a)$ for given $a \in A$ as the ath *component* of X. It is then natural to seek an analogue of the dot product of two vectors for subsets, and a natural proposal is that we form the sum of all the products of corresponding components of two subsets.

Now, calculating the products of corresponding components of two subsets X and Y of A is easy enough: if κ and λ are the respective characteristic functions of X and Y, then the ath such product is $\kappa(a)\ \lambda(a)$. This formula defines the characteristic function $\kappa\lambda$ of the intersection $X \cap Y$. The sum of such products is already defined when A is finite, and it turns out to be 0 when $X \cap Y$ is empty, or 1 when $X \cap Y$ is not empty. The natural thing to do would thus seem to be to use this result to extend the definition to the case where A is infinite. So we define the *dot product* $X \cdot Y$ of X and Y (subsets of A) to be 0 if $X \cap Y = \emptyset$ or 1 if $X \cap Y \neq \emptyset$. Dot product distributes over sums (unions) and is commutative. To make it really seem like an inner product, we ought to invent multiplication by "scalars" 0 and 1, but that is best left as an exercise. We may construe disjointness of sets as the analogue of orthogonality of vectors since $X \cdot Y = 0$ if and only if $X \cap Y = \emptyset$.

6. We may now interpet the result of applying a relation R in A into B to an element or a subset of A as analogous to the application of a matrix to a vector. If $X \subseteq A$, then X has components indexed by elements of A. Further, if $R \subseteq A \times B$, then R_M has its rows indexed by elements of A, and it makes sense to calculate the dot product of X with each of these subsets. Since there is a column of R for each element of B, this gives us a way of assigning a 0 or a 1 for each element of B; i.e., it gives us the characteristic function of a subset of B. This is simply formal multiplication (on the right) of the row-vector X by the matrix R_M.

Let us examine what subset Y of B is determined in this way. For each $b \in B$, the above process gives us either a 0 or a 1. We get a 1 if and only if for some $a \in A$, the ath component of X is 1 and the ath component of the bth column of R_M is 1. That is, we get a 1 if and only if there is an element a of X such that $(a, b) \in R$. Thus the subset Y of B determined as above is precisely the set usually denoted by $R[X]$ of all the "R-images of elements of X."

Note that R is somewhat "linear" in that it distributes over sums (unions). (It also behaves properly with respect to the scalar multiplication whose discovery was left as an exercise.) Relations thus play essentially the role of linear transformations. Not only this, but composition of relations can be thought of as matrix multiplication. If R is in A into B and S is in B into C, then the *composition* $S \circ R$ of R and S is a relation in A into C consisting of precisely those ordered pairs (a, c) with $a \in A$ and $c \in C$ for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Interpretation of these conditions in terms of the 1's in R_M and S_M easily shows that the matrix of $S \circ R$ is the matrix product $R_M S_M$.

This is a good point at which to notice that the reverse of $S \circ R$ is $\hat{R} \circ \hat{S}$, corresponding—as it must—to the fact that the transpose of a matrix product is the product of the transposes in the reverse order.

7. Let us note some other properties of binary relations that can be interpreted in terms familiar to anyone accustomed to matrices. Let M be the matrix of relation R in A into B. To say that R is on A is to say that every row of M

has at least one 1 in it. Similarly, R is onto B if and only if every column of M has at least one 1. On the other hand, R is single-valued if and only if every row of M has at *most* one 1. Thus R is a function on A in the usual sense if and only if each row of M has exactly one 1. Of course $\hat{R}$ is single-valued if and only if every column of M has at most one 1. This means that R is a one-to-one correspondence on A onto B if and only if every row and every column of M has exactly one 1.

The union of two binary relations R and S in A into B is again a binary relation in A into B. This allows us to make use of matrix addition, since it is clear that if M and N are the respective matrices of R and S then $M+N$ is the matrix of $R \cup S$. We then have "for free" the formulas $R\text{o}(S\cup T)=(R\text{o}S)\cup(R\text{o}T)$ and $(R\cup S)\text{o}T=(R\text{o}T)\cup(S\text{o}T)$.

The intersection of such an R with such an S will also be a binary relation in R into S, but the matrix analogue is the "element-wise product" of M with N obtained by multiplying corresponding elements. Since this article is strictly elementary and because few elementary courses in matrices use this sort of product, we drop the matter here.

8. We now consider a further extension of the preceding relation-theoretic terms. Let $\mathfrak{C}$ be a collection of subsets of A. We shall say that $\mathfrak{C}$ is orthogonal if the elements of $\mathfrak{C}$ are pairwise disjoint. (Hence for example, if $\mathfrak{C}$ has the finite intersection property no two elements of $\mathfrak{C}$ are orthogonal.) Regarding each element of $\mathfrak{C}$ as a "vector" a concept of linear independence can be formulated as follows: $C \notin \mathfrak{C}$ is *linearly independent* of $\mathfrak{C}$ provided that C is not the union of any subcollection of $\mathfrak{C}$. Consider this idea in connection with a relation R on A onto B. If $C \in \mathfrak{C}$ is dependent, i.e., if C is the union of other elements of $\mathfrak{C}$, then clearly $R[C]$ is dependent in $R(\mathfrak{C})=\{R[C] \mid C\in\mathfrak{C}\}$, and, conversely, if R is single-valued (i.e., each column of R_M contains exactly one 1), then R preserves linear independence as well.

What happens if R is in (but not on) A onto B is worth examining. Without loss of generality we may assume that the first row of the corresponding matrix R_M consists of 0's only. We show that for such R linear independence is never preserved, i.e., that X, Y, Z exist such that $X' \neq Y\cup Z$ but $R[X]=R[Y]\cup R[Z]$. Let $X=(0, 1, 1, \cdots)$, $Y=(1, 1, 1, \cdots)$, $Z=Y$. Clearly $X\neq Y\cup Z$. Then $R[X]=X\cdot R_M=(0, \cdots)$ where the coordinates following the initial 0 are dependent upon the form of the matrix R_M. Similarly $R[Y]=Y\cdot R_M=(0, \cdots)$ and $R[Z]=Z\cdot R_M=(0, \cdots)$. Since X, Y, and Z have the same entries in all coordinates but the first, $R[X]$, $R[Y]$ and $R[Z]$ will also be the same from the second coordinate on. Thus $R[X]=R[Y]\cup R[Z]$.

Consider now the case where R is a function on A onto B with $\hat{R}$ not single-valued. We may make the assumption that R_M has a 1 in row 1, column 1 and in row 2, column 1. By hypothesis every row contains at most one 1. Let X, Y, Z be as above. Then $X\neq Y\cup Z$ but clearly $R[X]=R[Y]\cup R[Z]$. Hence no such R preserves linear dependence. The requirement that R be single-valued is necessary, for if R is not a function, the following matrix

$$R_M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \cdot \end{pmatrix}$$

(with 1 along the off-diagonal and 0 elsewhere) will always give $R[X] \neq R[Y] \cup R[Z]$ whenever $X \neq Y \cup Z$.

It is of interest to note that if we define linear independence of a collection $\mathcal{C}$ by: $\mathcal{C}$ is linearly independent if and only if every element of $\mathcal{C}$ is linearly independent of the other elements of $\mathcal{C}$, then R preserves this linear independence provided that $\hat{R}$ is single-valued, and R always takes dependent subsets of A to dependent subsets of B. If $\hat{R}$ is single-valued then orthogonality is also preserved.

9. Let F be a relation on A onto B. An equivalence relation R on A can be defined in the following way: For $x, x' \in A$, we say $(x, x') \in R$ if and only if $F(x) = F(x')$. Similarly, an equivalence relation S can be defined on B by: $(y, y') \in S$ if and only if $F(y) = F(y')$ for $y, y' \in B$. Let A/R and B/S be the quotient sets whose elements are the R and S equivalence classes respectively. The following diagram illustrates all the relations considered:

$$\begin{tikzcd} & B/S \arrow[dr, "F_3"] & \\ A \arrow[ur, "F_4"] \arrow[rr, "F"] \arrow[dr, "F_1"'] & & B. \\ & A/R \arrow[ur, "F_2"'] & \end{tikzcd}$$

From the way A/R and B/S are defined, F_1 and $\hat{F}_3$ are single valued, where $F_1(x) = [x]_R$, $[x]_R = \{x' \in A \mid (x, x') \in R\}$,$([x]_R, y) \in F_2$ iff $(x, y) \in F$ and $(x, [y]_s) \in F_4$ iff $(x, y) \in F$. Clearly

$$F_M = F_{1_M} \circ F_{2_M} \quad \text{and} \quad F_M = F_{3_M} \circ F_{4_M}.$$

If we use F_1 and $\hat{F}_3$ to give A/R and B/S the quotient topology, then the following properties are not difficult to check (where A and B are topological spaces):

F is open (closed)$\Leftrightarrow F_4$ is open (closed),
F is lower semicontinuous (lsc)$\Leftrightarrow F_2$ is lsc,
F is upper semicontinuous (usc)$\Leftrightarrow F_2$ is usc,
F is continuous$\Leftrightarrow F_2$ is continuous.

One can further define a relation H on A/R onto B/S by $H = F_4 \circ \hat{F}_1$. It follows from the definition of the relations in the diagram that $H = \hat{F}_3 \circ F_2$. Hence

H is closed (open)$\Leftrightarrow F$ is closed (open)
H is lsc (usc, cont.)$\Leftrightarrow F$ is lsc (usc, cont.) and

$$H_M = F_{1_M} \circ \hat{F}_{4_M} = F_{2_M} \circ \hat{F}_{3_M}.$$

Since $\hat{F}_{1_M}$ is the transpose of F_{1_M}, and F_1 is single-valued, each column of $\hat{F}_{1_M}$ will contain but one 1 and similarly $\hat{F}_{3_M}$ will have only one 1 in each row. In matrix terminology we could say that the two matrices are elementary; thus the factorization of H_M results in the matrix equivalence of F_M and H_M.

Reference

1. E. Schröder, Vorlesungen über die Algebra der Logik, vol. 3, 2nd ed., Chelsea, New York, 1966.

RELATIVELY PRIME AMICABLE NUMBERS OF OPPOSITE PARITY

PETER HAGIS, JR., Temple University

1. Introduction. Two positive integers m and n are said to be amicable if

$$\sigma(m) = m + n = \sigma(n) \tag{1}$$

where, as usual, $\sigma(k)$ denotes the sum of the positive divisors of k. To date some 600 pairs of amicable numbers m and n have been found (see [**1**], [**2**], and [**4**]) all of which satisfy the conditions $m \equiv n \pmod 2$ and $(m, n) > 1$. The purpose of the present paper is to establish some necessary conditions for two relatively prime positive integers of opposite parity to be amicable.

Thus, in what follows we assume that

$$m = 2^r \prod_{i=1}^{s} p_i^{a_i}, \qquad n = \prod_{j=1}^{t} q_j^{b_j} \tag{2}$$

are amicable numbers such that the odd primes p_i and q_j are distinct for all i and j.

2. Some preliminary results. We begin by stating two lemmas. The proof of the first is given on page 34 of [**5**]. The second follows easily from Theorem 22 on page 37 of [**5**].

LEMMA 1. *If* $k \mid (A^2+B^2)$ *where* $k \geqq 2$, *and* $(A, B) = 1$, *then there exist integers* u *and* v *such that* $k = u^2+v^2$.

LEMMA 2. *If* $k \mid (A^2+2B^2)$ *where* $k \geqq 3$, *and* $(A, 2B) = 1$, *then there exist integers* u *and* v *such that* $k = u^2+2v^2$.

From (1) and (2) we have

$$\begin{aligned} m + n &= (2^{r+1} - 1) \prod_{i=1}^{s} (1 + p_i + p_i^2 + \cdots + p_i^{a_i}) \\ &= \prod_{j=1}^{t} (1 + q_j + \cdots + q_j^{b_j}). \end{aligned} \tag{3}$$

Since $m+n$ is odd we see immediately that $a_i \equiv b_j \equiv 0 \pmod 2$, so that $m = 2^r M^2$, $n = N^2$. By a result due to Kanold [**3**], $r = 1$. For completeness we include the proof.

Since $\hat{F}_{1_M}$ is the transpose of F_{1_M}, and F_1 is single-valued, each column of $\hat{F}_{1_M}$ will contain but one 1 and similarly $\hat{F}_{3_M}$ will have only one 1 in each row. In matrix terminology we could say that the two matrices are elementary; thus the factorization of H_M results in the matrix equivalence of F_M and H_M.

Reference

1. E. Schröder, Vorlesungen über die Algebra der Logik, vol. 3, 2nd ed., Chelsea, New York, 1966.

RELATIVELY PRIME AMICABLE NUMBERS OF OPPOSITE PARITY

PETER HAGIS, JR., Temple University

1. Introduction. Two positive integers m and n are said to be amicable if

$$\sigma(m) = m + n = \sigma(n) \tag{1}$$

where, as usual, $\sigma(k)$ denotes the sum of the positive divisors of k. To date some 600 pairs of amicable numbers m and n have been found (see [**1**], [**2**], and [**4**]) all of which satisfy the conditions $m \equiv n \pmod 2$ and $(m, n) > 1$. The purpose of the present paper is to establish some necessary conditions for two relatively prime positive integers of opposite parity to be amicable.

Thus, in what follows we assume that

$$m = 2^r \prod_{i=1}^{s} p_i^{a_i}, \qquad n = \prod_{j=1}^{t} q_j^{b_j} \tag{2}$$

are amicable numbers such that the odd primes p_i and q_j are distinct for all i and j.

2. Some preliminary results. We begin by stating two lemmas. The proof of the first is given on page 34 of [**5**]. The second follows easily from Theorem 22 on page 37 of [**5**].

LEMMA 1. *If $k \mid (A^2+B^2)$ where $k \geqq 2$, and $(A, B) = 1$, then there exist integers u and v such that $k = u^2+v^2$.*

LEMMA 2. *If $k \mid (A^2+2B^2)$ where $k \geqq 3$, and $(A, 2B) = 1$, then there exist integers u and v such that $k = u^2+2v^2$.*

From (1) and (2) we have

$$\begin{aligned} m + n &= (2^{r+1} - 1) \prod_{i=1}^{s}(1 + p_i + p_i^2 + \cdots + p_i^{a_i}) \\ &= \prod_{j=1}^{t}(1 + q_j + \cdots + q_j^{b_j}). \end{aligned} \tag{3}$$

Since $m+n$ is odd we see immediately that $a_i \equiv b_j \equiv 0 \pmod 2$, so that $m = 2^r M^2$, $n = N^2$. By a result due to Kanold [**3**], $r = 1$. For completeness we include the proof.

THEOREM 1 (Kanold). *If $2|m$ and m and n are relatively prime amicable numbers then $m=2M^2$ and $n=N^2$ where M and N are odd integers greater than 1.*

Proof. We already know that $m=2^rM^2$ and $n=N^2$. We shall show that $r=1$. Assume first that r is even. Then $2^{r+1}-1\equiv 3 \pmod 4$ and therefore $2^{r+1}-1$ has a prime factor P such that $P=4k+3$. Since, from (3), $P|(m+n)$ and since $m+n$ is the sum of two relatively prime squares, it follows from Lemma 1 that P is the sum of two squares. But this is impossible since $P\not\equiv 1 \pmod 4$. Therefore, r is odd.

If r is odd and $r>1$, then $2^{r+1}-1\equiv -1 \pmod 8$. It follows that $2^{r+1}-1$ is divisible by a prime Q such that $Q=8k+5$ or $Q=8k+7$. Also, $m+n=2C^2+N^2$ where $(2C, N)=1$. Since $Q|(m+n)$ it follows from Lemma 2 that $Q=u^2+2v^2$. Since $u^2+2v^2\not\equiv 5, 7 \pmod 8$ we have a contradiction. We conclude that $r=1$, and it is now obvious that neither M nor N is 1.

COROLLARY 1.1. $3\nmid mn$.

Proof. Since $r=1$ we see from (3) that $m+n\equiv 0 \pmod 3$. Therefore if $3|mn$ then $3|m$ *and* $3|n$ which is impossible since $(m, n)=1$.

COROLLARY 1.2. *If $p^{2a}\nmid mn$, then*
(i) *if $p=8k+1$, then $a\equiv 0, 1 \pmod 4$;*
(ii) *if $p=8k+3$, then $a\equiv 0, 2 \pmod 4$;*
(iii) *if $p=8k+5$, then $a\equiv 0, 3 \pmod 4$.*

Proof. From Theorem 1, (3), Lemma 2, and the fact that $u^2+2v^2\equiv 1, 3 \pmod 8$ if u is odd we see that $\sigma(p^{2a})\equiv 1, 3 \pmod 8$.

(i) If $p=8k+1$, then $\sigma(p^{2a})\equiv 1+p+p^2+\cdots+p^{2a}\equiv 1+2a \pmod 8$. Therefore, $1+2a\equiv 1, 3 \pmod 8$, and it follows that $a\equiv 0, 1 \pmod 4$.

(ii) If $p=8k+3$, then $\sigma(p^{2a})\equiv 1+3+1+3+1+\cdots+3+1\equiv 1+4a \pmod 8$. Therefore, $1+4a\equiv 1, 3 \pmod 8$ and $a\equiv 0, 2 \pmod 4$.

(iii) If $p=8k+5$, then $\sigma(p^{2a})\equiv 1+5+1+\cdots+5+1\equiv 1+6a \pmod 8$. Therefore, $1+6a\equiv 1, 3 \pmod 8$ and $a\equiv 0, 3 \pmod 4$.

COROLLARY 1.3. $m<2n-6$.

Proof. From (1) we have $m+n=\sigma(m)>m+(m/2)+2+1$.

3. Lower bounds for $\sum_{p|mn} 1/p$. In this section we shall establish lower bounds for $\sum_{p|mn} 1/p$ and immediately deduce therefrom lower bounds for the number of distinct prime divisors of mn and the magnitude of mn. In what follows p, P, q, Q will always denote primes and a sum such as $\frac{1}{5}+\frac{1}{7}+\frac{1}{11}$ will be denoted by $\sum_{p=5}^{11} 1/p$. We first prove two lemmas.

LEMMA 3. *Let m and n be relatively prime amicable numbers of opposite parity such that $2|m$. If $mn=2\prod_{i=1}^{k} p_i^{a_i}$ where $p_1<p_2<\cdots<p_k$, and Q is a prime such that $p_J<Q\leqq p_{J+1}$ $(1\leqq J<k)$, then*

$$\log\frac{8}{3}<\log\prod_{i=1}^{J}\{p_i/(p_i-1)\}+Q\left(\sum_{i=1}^{k}1/p_i-\sum_{i=1}^{J}1/p_i\right)\log(Q/(Q-1)).$$

Proof. Omitting the subscripts in (2) we have from (1) $1+n/m=\sigma(m)/m$ $=\frac{3}{2}\prod\{(p-p^{-a})/(p-1)\}<\frac{3}{2}\prod\{p/(p-1)\}$, and $1+m/n=\sigma(n)/n=\prod\{(q-q^{-b})$ $/(q-1)\}<\prod\{q/(q-1)\}$. Multiplying we have

$$2+n/m+m/n<\frac{3}{2}\prod_{i=1}^{k}\{p_i/(p_i-1)\}. \tag{4}$$

But $n/m+m/n>2$. Therefore, $\frac{8}{3}<\prod_{i=1}^{k}(1-1/p_i)^{-1}$. Taking logarithms we have

$$\begin{aligned}\log\frac{8}{3} &< \log\prod_{i=1}^{J}\{p_i/(p_i-1)\}+\sum_{i=J+1}^{k}\sum_{j=1}^{\infty}1/jp_i^j\\ &\leqq \log\prod_{i=1}^{J}\{p_i/(p_i-1)\}+\sum_{i=J+1}^{k}\sum_{j=1}^{\infty}1/jp_iQ^{j-1}\\ &= \log\prod_{i=1}^{J}\{p_i/(p_i-1)\}+\sum_{iJ=+1}^{k}1/p_i\sum_{j=1}^{\infty}Q/jQ^{\cdot}\\ &= \log\prod_{i=1}^{J}\{p_i/(p_i-1)\}+Q\log\frac{Q}{Q-1}\cdot\sum_{i=J+1}^{k}1/p_i.\end{aligned}$$

The necessary modifications in both the statement and proof of this lemma in case $Q\leqq p_1$ or $p_k<Q$ are obvious and are therefore omitted.

LEMMA 4. *The function* $f(x)=x\log\{x/(x-1)\}$ *is monotonic decreasing on the interval* $[2,\infty)$.

Proof. $f'(x)=\log\{1+1/(x-1)\}-1/(x-1)$. Since $\log(1+X)<X$ if $0<X\leqq 1$ we see that $f'(x)<0$ if $x\geqq 2$.

We are now prepared to establish lower bounds for $\sum_{p|mn}1/p$. From Lemma 3 we have

$$\sum_{p|mn}1/p>\frac{1}{2}+\sum_{i=1}^{J}1/p_i+\frac{\log\left\{\frac{8}{3}\prod_{i=1}^{J}(p_i-1)/p_i\right\}}{Q\log\{Q/(Q-1)\}}, \tag{5}$$

while it follows easily from Lemma 4 that if q is a prime less than Q then

$$1/q+\frac{\log\{(q-1)/q\}}{Q\log\{Q/(Q-1)\}}<0. \tag{6}$$

If $5|mn$ then $p_1=5$ (recall Corollary 1.1). If P is the greatest prime less than Q then it follows from (5) and (6) that

$$\sum_{p|mn}1/p>\frac{1}{2}+\sum_{p=5}^{P}1/p+\frac{\log\left\{\frac{8}{3}\prod_{p=5}^{P}(p-1)/p\right\}}{Q\log\{Q/(Q-1)\}}=F(Q). \tag{7}$$

Our task is now to determine Q so that the right hand member of (7) is maximal. Since an analytic attack seemed difficult a program was written for the

CDC6400 in order to study the behavior of $F(Q)$. It was found that $F(7) < F(11) < \cdots < F(73) < F(79)$ while $F(79) > F(83)$.

If $5 \nmid mn$ then $p_1 \geqq 7$. If P is the greatest prime less than Q then from (5) and (6)

$$\sum_{p|mn} 1/p > \frac{1}{2} + \sum_{p=7}^{P} 1/p + \frac{\log\left\{\frac{8}{3}\prod_{p=7}^{P}(p-1)/p\right\}}{Q\log\{Q/(Q-1)\}} = G(Q).$$

Making use of the CDC6400 it was found that $G(Q)$ is monotonic increasing between $Q=11$ and $Q=257$ while $G(263) < G(257)$.

We collect these results in the following

THEOREM 2. *If m and n are relatively prime amicable numbers of opposite parity, then*

(i) *if* $5 | mn$,

$$\sum_{p|mn} 1/p > \frac{1}{2} + \sum_{p=5}^{73} 1/p + \frac{\log\left\{4\prod_{p=3}^{73}(p-1)/p\right\}}{79\log\{79/78\}}$$
$$\doteq .5 + .923232 + .008288 > 1.43151;$$

(ii) *if* $5 \nmid mn$,

$$\sum_{p|mn} 1/p > \frac{1}{2} + \sum_{p=7}^{251} 1/p + \frac{\log\left\{5\sum_{p=3}^{251}(p-1)/p\right\}}{257\log\{257/256\}}$$
$$\doteq .5 + .950306 + .003520 > 1.45382.$$

COROLLARY 2.1. *mn is divisible by at least 21 different primes. If $5 \nmid mn$ then mn is divisible by at least 53 primes.*

COROLLARY 2.2. $mn > 10^{74}$. *If $5 \nmid mn$ then* $mn > 10^{238}$.

Proof. If $5 | mn$ then $5^6 | mn$ by Corollary 1.2. The smallest number of the form $2\cdot 5^6\prod_{i=1}^{19} p_i^{a_i}$, where $p_i > 3$, $a_i = 2$ if $p_i \equiv \pm 1 \pmod 8$, $a_i = 4$ if $p_i \equiv 3 \pmod 8$, and $a_i = 6$ if $p_i \equiv 5 \pmod 8$, is easily seen to be

$$K = 2\cdot 5^6\cdot 11^4(7\cdot 17\cdots 167\cdot 191)^2 = 2\cdot 5^6\cdot 11^4\cdot R^2$$

where R is the product of the 18 primes between 7 and 191, inclusive, which are congruent to 1 or 7 modulo 8. Taking the logarithm of K to the base 10 we find that $K > 10^{74}$. From Corollaries 1.1, 1.2, 2.1 and Theorem 1 we see that if $5 | mn$, then $mn \geqq K$.

If $5 \nmid mn$ then the smallest odd prime dividing mn is not less than 7. The smallest number of the form $2\sum_{i=1}^{52} p_i^{a_i}$, where $p_1 \geqq 7$, $a_i = 2$ if $p_i \equiv \pm 1 \pmod 8$, $a_i = 4$ if $p_i \equiv 3 \pmod 8$, and $a_i = 6$ if $p_i \equiv 5 \pmod 8$ is

$$L = 2\cdot 11^4\cdot 19^4(7\cdot 17\cdots 577\cdot 593)^2 = 2\cdot 11^4\cdot 19^4\cdot S^2$$

where S is the product of the 50 primes which are congruent to 1 or 7 modulo 8 and lie between 7 and 593 inclusive. Since $\log_{10} L \doteq 238.524$, and since $mn \geqq L$ by Corollaries 1.1, 1.2, 2.1 and Theorem 1 we see that $mn > 10^{238}$.

COROLLARY 2.3. *$m > 5 \cdot 10^{36}$ and $n > 7 \cdot 10^{36}$.*

Proof. From Corollaries 1.3 and 2.2 we have $2n^2 > mn > 10^{74}$ from which it follows that $n > 7 \cdot 10^{36}$. If $m > n$ then $m > 7 \cdot 10^{36}$ also. If $m < n$ there are two possibilities. If $3m > n$ then $3m^2 > mn > 10^{74}$ and therefore $m > 5 \cdot 10^{36}$. If $3m < n$ then from (1), (2), and Theorem 1, $1.5 \prod_{i=1}^{s} p/(p-1) > \sigma(m)/m = (m+n)/m > 4$. Therefore, if P is the smallest prime such that $\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdots P/(P-1) > 4$ then certainly $m > 2 \cdot (5 \cdot 7 \cdots P)^2$. Making use of the CDC6400 it was found that $P = 79$ and $m > 10^{59}$.

4. Upper bounds for $\sum_{p|mn} 1/p$. If $m > n$ then it is possible to establish upper bounds for $\sum_{p|mn} 1/p$. Thus, using the same notation as in Lemma 3, we have

$$\begin{aligned} 2 + n/m + m/n &= \frac{3}{2} \prod_{i=1}^{k} \{(p_i - p_i^{-a_i})/(p_i - 1)\} \\ &= \frac{3}{2} \prod_{i=1}^{k} (1 - 1/p_i)^{-1} \prod_{i=1}^{k} (1 - 1/p_i^{a_i+1}). \end{aligned}$$

Taking logarithms we have

$$\log(2 + n/m + m/n) = \log \frac{3}{2} + \sum_{i=1}^{k} 1/p_i + \sum_{i=1}^{k} \sum_{j=1}^{\infty} \left(\frac{1}{(j+1)p_i^{j+1}} - \frac{1}{j p_i^{(a_i+1)j}} \right). \tag{8}$$

Since $a_i \geqq 2$, every term in the last summation in the right hand member is positive. Also, since $2 > m/n$ from Corollary 1.3, and since $x + x^{-1}$ is an increasing function for $x > 1$ we see that $2 + n/m + m/n < 9/2$.

If $5 | mn$ then $p_1 = 5$, and $a_1 \geqq 6$ from Corollary 1.2. It then follows from (8) and the preceding remarks that

$$\log \frac{9}{2} > -\frac{1}{2} - \frac{1}{5} + \log \frac{3}{2} - \log(1 - 1/5) + \log(1 - 1/5^7) + \sum_{p|mn} 1/p,$$

or $\sum_{p|mn} 1/p < \frac{7}{10} + \log 46875/19531$.

If $5 \nmid mn$, then from (8) and the succeeding remarks, $\log \frac{9}{2} > -\frac{1}{2} + \log \frac{3}{2} + \sum_{p|mn} 1/p$, or $\sum_{p|mn} 1/p < \frac{1}{2} + \log 3$.

We state these results as

THEOREM 3. *If m and n are relatively prime amicable numbers of opposite parity such that $2 | m$ and $m > n$, then*

(i) if $5 | mn$, $\displaystyle\sum_{p|mn} 1/p < \frac{7}{10} + \log \frac{46875}{19531} < 1.57549;$

(ii) if $5 \nmid mn$, $\displaystyle\sum_{p|mn} 1/p < \frac{1}{2} + \log 3 < 1.59862.$

Remark. The bounds established in Theorem 3 also hold if $m<n$ and $2>n/m$.

5. Bounds on $mn/\phi(mn)$. From (4) it follows that

$$\frac{4}{3}(2+n/m+m/n) < \prod_{p\mid mn}\{p/(p-1)\} = mn/\phi(mn) \tag{9}$$

where $\phi(k)$ is the Euler phi-function. We shall now establish an upper bound for $mn/\phi(mn)$.

First, from (1) we see that $m/n+1=\sigma(n)/n=\sum_{d\mid n}1/d$, $n/m+1=\sigma(m)/m=\sum_{d\mid m}1/d$. Adding, and then multiplying, we have

$$\sum_{d\mid mn}1/d = \sigma(mn)/mn = 2+m/n+n/m = \sum_{d\mid m}1/d+\sum_{d\mid n}1/d. \tag{10}$$

Now,

$$mn/\phi(mn) = \prod_{p\mid mn}(1-1/p)^{-1} = \prod_{p\mid mn}(1+1/p+1/p^2)(1-1/p^3)^{-1}. \tag{11}$$

If $5\mid mn$ then since $3\nmid mn$, $2\|mn$, and $5^6\mid mn$ we have

$$\begin{aligned} mn/\phi(mn) &< \frac{26}{27}\prod_{p}(1-1/p^3)^{-1}\prod_{p\mid mn}(1+1/p+1/p^2) \\ &= \frac{26}{27}(1+1/2+1/4)(1+1/2)^{-1} \\ &\quad (1+1/5+1/25)(1+1/5+\cdots+1/5^6)^{-1} \\ &\quad \cdot\zeta(3)(1+1/2)(1+1/5+\cdots+1/5^6)\prod_{p\mid mn}{}^{*}(1+1/p+1/p^2). \end{aligned}$$

$\prod^{*}$ indicates that the primes 2 and 5 are to be omitted, and $\zeta(s)$ is the Riemann zeta function. From Theorem 1 it now follows that

$$mn/\phi(mn) < \frac{1763125}{1582011}\zeta(3)\sum_{d\mid mn}1/d.$$

From (10)

$$mn/\phi(mn) < \frac{1763125}{1582011}\zeta(3)(2+n/m+m/n).$$

If $5\nmid mn$ then from (11)

$$\begin{aligned} mn/\phi(mn) &< \frac{26}{27}\cdot\frac{124}{125}\prod_{p}(1-1/p^3)^{-1}\prod_{p\mid mn}(1+1/p+1/p^2) \\ &= \frac{26}{27}\cdot\frac{124}{125}\cdot\frac{7}{4}\cdot\frac{2}{3}\zeta(3)(1+1/2)\prod_{p\mid mn}{}'(1+1/p+1/p^2) \end{aligned}$$

where $\prod'$ indicates the omission of 2. From Theorem 1 and (10)

$$mn/\phi(mn) < \frac{11284}{10125}\zeta(3)\sum_{d|mn} 1/d = \frac{11284}{10125}\zeta(3)(2 + n/m + m/n).$$

We collect these results as

THEOREM 4. *If m and n are relatively prime amicable numbers of opposite parity, then*

(i) *if* $5|mn$,

$$\frac{4}{3}(2 + n/m + m/n) < mn/\phi(mn) < \frac{1763125}{1582011}\zeta(3)(2 + n/m + m/n)$$
$$< 1.33967(2 + n/m + m/n);$$

(ii) *if* $5\nmid mn$,

$$\frac{4}{3}(2 + n/m + m/n) < mn/\phi(mn) < \frac{11284}{10125}\zeta(3)(2 + n/m + m/n)$$
$$< 1.33965(2 + n/m + m/n).$$

COROLLARY 4.1. $.746 < \phi(mn)\sigma(mn)/(mn)^2 < .75$.

Since $n/m + m/n > 2$ and since $n/m + m/n < 2.5$ if $m > n$ (recall Corollary 1.3) we have

COROLLARY 4.2. *If $m > n$, then*

$$16/3 < mn/\phi(mn) < 6.0286.$$

References

1. E. B. Escott, Amicable numbers, Scripta Math., 12 (1946) 61–72.
2. M. Garcia, New amicable pairs, Scripta Math., 23 (1957) 167–171.
3. H-J. Kanold, Über befreundete Zahlen II, Math. Nachr., 10 (1953) 99–111.
4. P. Poulet, 43 new couples of amicable numbers, Scripta Math., 14 (1948) 77.
5. H. Rademacher, Lectures on Elementary Number Theory, Blaisdell, New York, 1964.

AN IMPROVED SOLUTION TO "INSTANT INSANITY"

B. L. SCHWARTZ, The Mitre Corporation

1. Introduction. In Vol. 41, No. 4, of this magazine, T. A. Brown presented an ingenious method for solving the currently popular puzzle called *Instant Insanity* [1]. In this note, we shall describe and illustrate a slight improvement over his procedure.

Instant Insanity is a puzzle whose pieces are four cubes, illustrated in Figure 1. Each face is colored one of four colors, as shown, White, Red, Blue, or Green. The objective is to assemble the cubes in a $1 \times 1 \times 4$ square prism in which each of the four 1×4 lateral faces displays all four colors.

$$mn/\phi(mn) < \frac{11284}{10125}\zeta(3)\sum_{d|mn} 1/d = \frac{11284}{10125}\zeta(3)(2 + n/m + m/n).$$

We collect these results as

THEOREM 4. *If m and n are relatively prime amicable numbers of opposite parity, then*

(i) *if* $5|mn$,

$$\frac{4}{3}(2 + n/m + m/n) < mn/\phi(mn) < \frac{1763125}{1582011}\zeta(3)(2 + n/m + m/n)$$
$$< 1.33967(2 + n/m + m/n);$$

(ii) *if* $5\nmid mn$,

$$\frac{4}{3}(2 + n/m + m/n) < mn/\phi(mn) < \frac{11284}{10125}\zeta(3)(2 + n/m + m/n)$$
$$< 1.33965(2 + n/m + m/n).$$

COROLLARY 4.1. $.746 < \phi(mn)\sigma(mn)/(mn)^2 < .75$.

Since $n/m + m/n > 2$ and since $n/m + m/n < 2.5$ if $m > n$ (recall Corollary 1.3) we have

COROLLARY 4.2. *If $m > n$, then*

$$16/3 < mn/\phi(mn) < 6.0286.$$

References

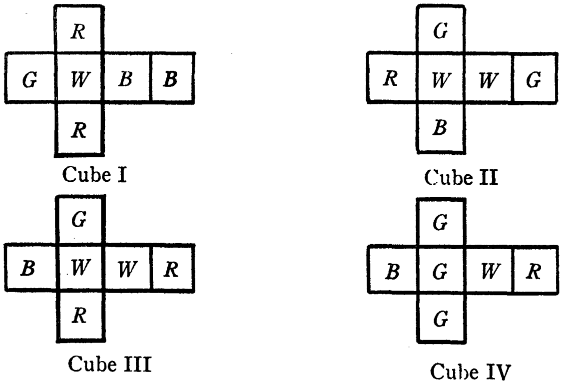
1. E. B. Escott, Amicable numbers, Scripta Math., 12 (1946) 61–72.
2. M. Garcia, New amicable pairs, Scripta Math., 23 (1957) 167–171.
3. H-J. Kanold, Über befreundete Zahlen II, Math. Nachr., 10 (1953) 99–111.
4. P. Poulet, 43 new couples of amicable numbers, Scripta Math., 14 (1948) 77.
5. H. Rademacher, Lectures on Elementary Number Theory, Blaisdell, New York, 1964.

AN IMPROVED SOLUTION TO "INSTANT INSANITY"

B. L. SCHWARTZ, The Mitre Corporation

1. Introduction. In Vol. 41, No. 4, of this magazine, T. A. Brown presented an ingenious method for solving the currently popular puzzle called *Instant Insanity* [1]. In this note, we shall describe and illustrate a slight improvement over his procedure.

Instant Insanity is a puzzle whose pieces are four cubes, illustrated in Figure 1. Each face is colored one of four colors, as shown, White, Red, Blue, or Green. The objective is to assemble the cubes in a $1\times1\times4$ square prism in which each of the four 1×4 lateral faces displays all four colors.



G = green face B = blue face
W = white face R = red face

FIG. 1.

2. Prior theory. Brown has shown the key concept that the colored faces may be considered in terms of opposite pairs of faces. For example, in Cube I, since B and G appear as one pair of opposite faces, it suffices to consider the pair BG.

This can be shown as follows. Let the cubes be assembled in any way so that those two faces occur on lateral faces of the prism. Then by an appropriate rotation of this cube thru 180°, the G and B faces can be interchanged, without disturbing the arrangement of any of the other colors on the lateral faces.

To utilize this observation Brown proposes the following: Each color is given a numerical value as follows:

$$R \approx 1$$
$$W \approx 2$$
$$B \approx 3$$
$$G \approx 5$$

Then one can associate an integer with each pair of opposite faces; viz., the pair is given a value equal to the product of the associated values of its two colors. Thus, the BG pair on Cube I gets a value of $(3)(5)=15$.

The requirement is then to choose a set of four pairs, one pair from each cube, so that the product of all the pair-faces numerical values is $900=30^2=(1\cdot 2\cdot 3\cdot 5)^2$. This assures that each color occurs exactly twice in the four chosen pairs. Hence, the cubes can be arranged in a square prism so that each color occurs exactly once on each of two opposite faces.

If this can be done in two distinct, nonintersecting ways, then all four rectangular faces can be arranged to meet the required condition; and this solves the puzzle.

3. Procedure. The discovery of the required sets of face-pairs is a trial-and-error process. However, it is a vastly simpler one than trial-and-error experimentation with the cubes without the use of the pairing principle. This is the value of Brown's discovery of the principle. He proposes to simplify the trials by using a tabular arrangement of the face-pair values of the cubes, as in Table 1.

TABLE 1. NUMERICAL FACE-PAIR VALUES

Face Pair	(1)	(2)	(3)
Cube I	1	6	15
Cube II	2	10	15
Cube III	2	5	6
Cube IV	5	6	25

4. Improvement. The present writer's proposal for improving the process is simply to dispense with the numerical values, and tabulate the face-pair colors themselves, as in Table 2. Now the objective becomes to find, by inspection, a pair set comprising one entry from each row, such that the (literal) product of the chosen entries is $B^2R^2W^2G^2$. Our experience has been that this is easier than the use of the numerical values.

TABLE 2. LITERAL FACE-PAIR VALUES

Face Pair	(1)	(2)	(3)
Cube I	*RR*	*BW*	*BG*
Cube II	*RW*	*WG*	*BG*
Cube III	*RW*	*RG*	*BW*
Cube IV	*RG*	*BW*	*GG*

5. Examples. A. Suppose in Table 2 that the solver has begun by considering in Cubes I and II the pairs I-3 and II-3, which happen both to be *BG* pairs. It is clear that the choices for Cubes III and IV must both omit any *B* or *G* factors. An immediate scan of the last two rows of the table shows that this is impossible for Cube IV. Hence, the starting selection is a blind alley and can be discarded.

B. Suppose again that the solver has begun with the pairs I-2 $=BW$ and II-3 $=BG$. Since no R appears in these choices, the remaining two pairs chosen must have 2 R's. No *RR* pair occurs on Cubes III or IV. Hence, each of the remaining two choices must include just one R face. But this immediately limits the choice on Cube IV to the IV-1 pair. Once this is chosen, it is immediate that the choice on Cube III of the III-1 pair completes one feasible set of four pairs.

6. Manual operation. For trial-and-error solution, simply manipulating the cubes in hand without pencil and paper, the writer has found the following "algorithm" practical. Line up the cubes in a row arbitrarily. Then:

1. Get one pair of opposite 1×4 lateral faces balanced, i.e., displaying (altogether) two of each color. This is pure trial-and-error, and is the hardest part of the solution.

2. Adjust that pair by 180° rotation of cubes until each of those two faces has one of each color. Now place the cubes with one of the "solved" faces down on the table, the other face uppermost. In further adjustment, rotate cubes only about vertical lines, keeping the solved face flat on the table.

3. By further trial-and-error, but using only the rotations described above, get the other pair of faces balanced. Rotations of 90° or 180° are permitted in this step.

4. Finally by 180° rotations only, reversing appropriate face pairs of the second set, complete the solution.

7. Quantitative considerations. Brown has shown that the original form of *Instant Insanity* has 82,944 essentially different cases to consider, of which only two are solutions to the puzzle. The plight of the trial-and-error solver (either random or systematic) is obvious. By use of the face-pairing principle, the number of cases is reduced to $3^4=81$, with two solutions. Brown has suggested a systematic method for examining those 81 cases. On the bottom of page 168, he gives two lists of nine partial products. Each number in the first list is to be matched with each in the second to see if the resulting complete product equals 900.

The present proposal now still further improves the effectiveness of the trial-and-error search in three ways.

First, it is sometimes possible to conclude testing of a possibility without forming the complete product. If we begin with the two pairs I-1 and II-1, with literal values RR and RW, we can stop immediately without further testing, since the partial product already contains three R factors, and is therefore infeasible.

Secondly, when such cases arise, they serve to eliminate several candidate pair sets among the 81, not just one. In the case above, all of the nine possible ways of completing the pair set by any choices from Cubes III and IV are rejected at a single stroke.

Finally, the inspection is performed more readily by the human problem solver when the formulation is in literal, rather than numerical, form. This is true whether or not the complete product is formed. It is easier to see at a glance whether $(BW)(WG)(RG)(BW)$ is equal to $R^2B^2W^2G^2$ than it is to see if $(3\cdot 2)(2\cdot 5)(1\cdot 5)(3\cdot 2)$ is 900.

The writer has found no convenient objective criterion by which the improvement due to these factors might be measured. His empirical experience in solving a number of cases suggests that the time spent using the literal method would be one-third to one-half that spent using the numerical. While this advantage is not so great as Brown's initial giant step, we still feel we have made a worthwhile additional contribution.

Reference

1. T. A. Brown, A note on "instant insanity," this MAGAZINE, 41 (1968) 167–69.

THE PACKING OF EQUAL CIRCLES IN A SQUARE

MICHAEL GOLDBERG, Washington, D. C.

1. Introduction. There are many interesting extremal problems associated with the packing of circles and spheres. An excellent summary of these problems is contained in the book by Fejes Tóth [1]. These include the packing of circles on the surface of the sphere which has been further investigated by the present author [2].

One of the oldest of packing problems is the packing of equal circles in a rectangle or a square. Its most common application today is the packing of bottles or cans in a box. In spite of its antiquity, and its common utility, very little has been done on the problem from an analytical standpoint. The best recent references are the papers by Schaer and Meir [3, 4]. They derived the "best" arrangements for the packing of n circles in a square for $n \leqq 9$. The term "best" implies the smallest square for the given n circles, or the largest diameter of the circles for a given square.

Some of the techniques employed by the author in finding solutions for the sphere are here used to find solution in a square. The sphere had no boundary conditions. The boundary of the square introduces further complications.

The results presented here should be compared with the results of Kravitz [5] on the packing of equal circles in a larger circle.

2. Notation. The problem is the same as maximizing the minimum distance between the pairs of n points in a unit square. If this distance is designated by m, then the edge e of the square enclosing n circles of diameter m, centered on these points, is given by $e=1+m$. Hence $d(n)$, the density of coverage of the latter square by the circles, is given by $d(n)=n(\pi m^2)/4(1+m)^2=(\pi/4)nm^2/(1+m)^2$.

Many efficient arrangements are obtained by packing in rows of equally spaced circles. The symbol $n(p, q, r, \cdots)$ indicates n circles arranged in rows so that p equally spaced circles are in the first row, q circles in the second row, etc. One may pack them closely in interlaced rows parallel to the edge of the square, as shown in the figures for $n=10$, 12, 13, etc.; or they may be packed parallel to the diagonal of the square, as shown in the figures for $n=9$, 16, 25, etc.

3. Stability of packing. For extremal solutions, it is necessary that a structure of circles must connect all the sides of the square. Each circle in the structure must make at least three contacts with other circles or with the sides of the square. These contacts cannot be limited to a semicircle, for then the circle could be moved to separate it from the structure. There may be, however, other circles which are not part of the structure. Such solutions appear for $n=7$, 19, 21, 22 and 26. The foregoing conditions are necessary for static stability and hence, for an extremal solution. However, for a given n, there may be several such stable arrangements with different packing densities. They correspond to local extrema. As an example, the arrangement of 20 circles in Table 2 and its figure, seems to be stable. Yet, 22 circles of the same size can be arranged in the square, as shown in Table 1 and its figure.

4. Description of the special arrangements. Some of the best results are not obtained as regular arrangements in equally spaced rows. Instead, some may be considered as combinations of several sets of the best arrangements for smaller numbers of circles. For example, for $n=13$, the arrangement may be considered as the combination of four sets of five circles in which some of the circles have been superimposed. For $n=18$, the arrangement may be considered as the combination of four sets of the six circles shown for $n=6$. Also, the efficient cluster of eight circles is visible in the upper right corner of the figures for $n=14$ and $n=15$, and in the upper right and lower left corners of the figure for $n=23$.

A dense arrangement of 20 circles consisting of five rows of four circles in each row will make a rectangle which is slightly longer than its height. Therefore, a lateral compression must be applied to shorten and heighten the rectangle to make it into a square. Two of the rows are shifted laterally, while the circles in each row remain in contact. The spacing between adjacent rows is increased. An interior circle, which had made six contacts with its neighboring circles in the dense arrangement, now makes only four contacts in the square arrangement.

On the other hand, the arrangement for $n=11$ has no symmetry and only a trace of regularity. It is obtained by the removal of a circle from the arrangement for 12 circles, and a readjustment of the remaining circles until stability is restored. Similarly, the arrangements for $n=15$ and $n=24$ are obtained by removal of a circle from $n=16$ and $n=25$, and collapsing the corresponding rows and columns. For $n=14$, 17 and 21, more complicated adjustments were required.

5. Tabulation of results. For the regular arrangements, it is possible to obtain exact solutions since they involve only linear or quadratic equations. In the cases of irregular arrangements, equations of higher degree are involved. For example, for $n=19$, the distance m is found from the equation $5\sqrt{m^2-1/16}+\sqrt{m^2-(1/4-m/2)^2}=1$. In these cases, the numerical results were obtained by successive approximation.

In many cases, there are several promising arrangements, and each must be investigated to determine the best. An analytical attack is lacking, and a rigorous demonstration of the attainment of the best result is still needed. The best results obtained are summarized in Table 1 and shown in the corresponding figures. Less efficient arrangements which have been tried are given in the upper part of Table 2 and its corresponding figures.

6. Larger values of *n*. As n increases without limit, the arrangement approaches *regular dense packing* in the plane. Then $d(n)$ approaches $\pi/\sqrt{12}=0.9069$ as a limit [1, p. 58]. Hence, $4d(n)/\pi=1.1547$ for $n\to\infty$. In this arrangement, each interior circle is surrounded and in contact with six other circles.

For *rectangular lattice packing*, each interior circle is surrounded and in contact with four other circles. Then, $4d(n)/\pi=1.000$ and this value is independent of n. This is shown in Table 1 for $n=1$, 4, 9, 16 and 25. For n25, $\leqq$these are seen to be the most efficient packings.

For $n=30$, it seems that $4d(n)/\pi$ may always be greater than unity. See the lower part of Table 2. As a particularly interesting example, a square of edge 14 can enclose 196 circles of unit diameter in rectangular lattice packing. Yet, a dense array of 16 rows, each row containing 13 circles (a total of 208 circles), can be enclosed in a rectangle of dimensions 13.50 by 13.98. Thus, a square of edge 14, which contains this rectangle, can enclose at least 208 circles, instead of only 196.

The shift method, used for $n=20$, was used also for the determination of the data for $n=30$, 42, 143, 168 and 340 in Table 2. It is illustrated in the figure for $n=42$.

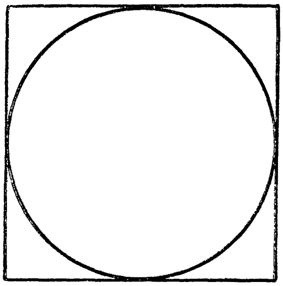
If a dense collection is enclosed in a rectangle which approximated a square, but in which the height is greater than the length, then a vertical compression will produce a square by reducing the height and spreading the circles in the rows. This method was used for obtaining the data for $n=39$, 52, 80, 99, 120, 161, 188, 270 and 304. This method was used also for $n=24(4,4,4,4,4,4)$ and is illustrated among the figures for Table 2.

If the rectangle enclosing the dense array is a close approximation of a square, then only a small distortion is needed to reshape it into a square. The greater the distortion, which may be a shift or a compression, the greater is the loss in efficiency. Hence, the function which expresses the density in terms of n, as seen in Tables 1 and 2, is not monotonic. The arrangements listed at the bottom of Table 2 were selected from the more efficient arrangements.

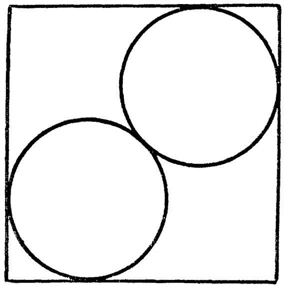
TABLE 1

n	Arrangement	m		$4d(n)/\pi$
2		$\sqrt{2}$	$=1.414$	.6850
3	1,2	$\sqrt{6}-\sqrt{2}$	$=1.035$	.7755
4	1,2,1	1	$=1.000$	1.0000
5	2,1,2	$\sqrt{2}/2$	$=.707$	.8580
6	2,2,2	$\sqrt{13}/6$	$=.601$	.8460
7	Irregular	$2(2-\sqrt{3})$	$=.536$	.8533
8	Based on 3 circles	$(\sqrt{6}-\sqrt{2})/2$	$=.518$	.9320
9	1,2,3,2,1	1/2	$=.500$	1.0000
10	2,3,2,3	5/12	$=.4167$	.8650
11	Irregular		$=.398$	.8921
12	3,3,3,3	$\sqrt{34}/15$	$=.389$	.9420
13	3,2,3,2,3	$\sqrt{2}/4$	$=.353$	.885
14	Modified 15	$(\sqrt{6}-\sqrt{2})/3$	$=.3451$	.921
15	Modified 16	$4/(8+\sqrt{6}+\sqrt{2})$	$=.3372$	.954
16	1,2,3,4,3,2,1	1/3	$=.3333$	1.000
17	Modified 18		$=.3045$	.926
18	4,3,4,3,4	$\sqrt{13}/12$	$=.3005$	.961
19	Modified 20		$=.290$	.960
20	4,4,4,4,4, shift	$3/8-\sqrt{2}/16$	$=.2866$	.992
21	Modified 22		$=.2704$	.952
22	Modified 25	$2-\sqrt{3}$	$=.2680$	.981
23	Based on 8 circles	$(\sqrt{6}-\sqrt{2})/4$	$=.2588$	.972
24	Modified 25	$1/(2+\sqrt{3/2}+1/\sqrt{2})$	$=.2543$	.987
25	1,2,3,4,5,4,3,2,1	1/4	$=.2500$	1.000
26	Modified 27		$=.2373$	.957
27	5,4,5,4,5,4	$\sqrt{89}/40$	$=.2358$	.983

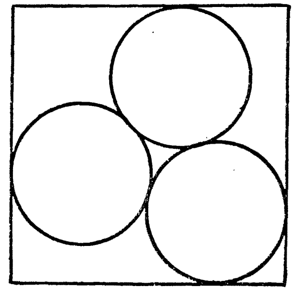
FIGURES FOR TABLE 1



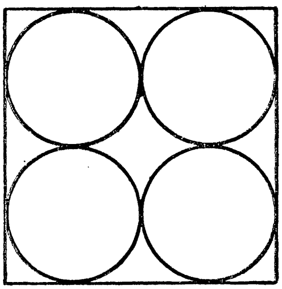
1



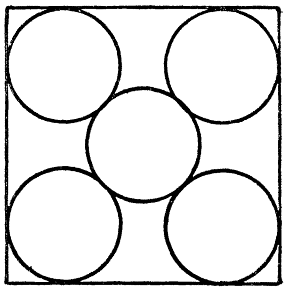
2



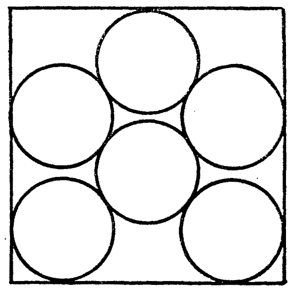
3



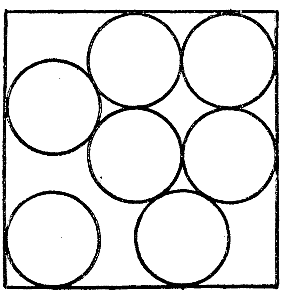
4



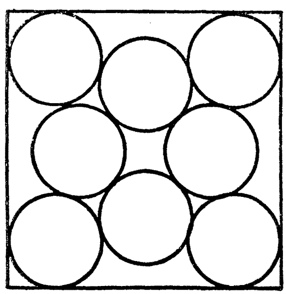
5



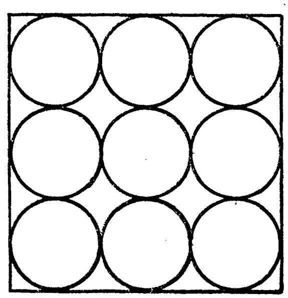
6



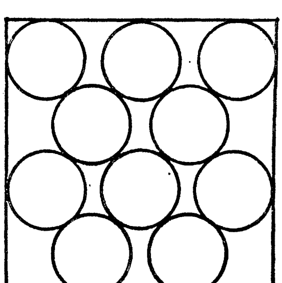
7



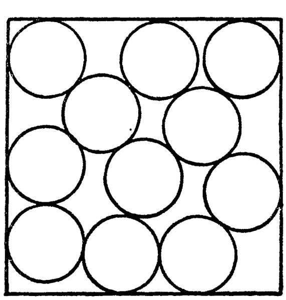
8



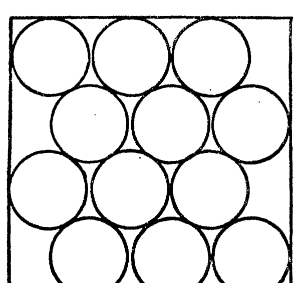
9



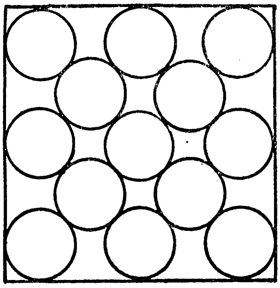
10

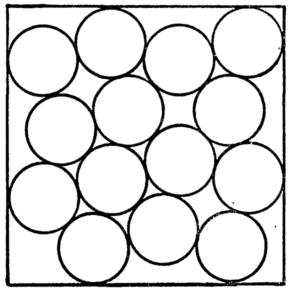


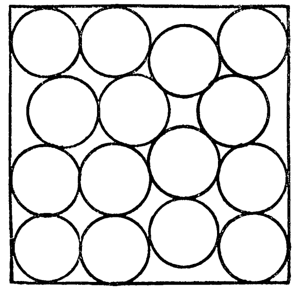
11

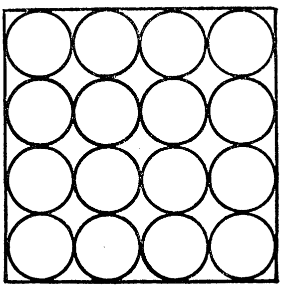


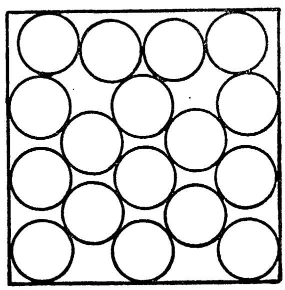
12

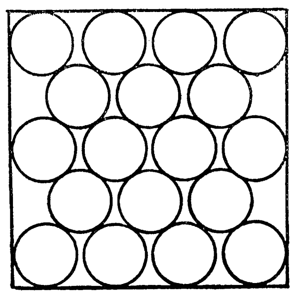

13

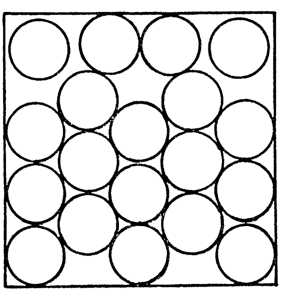

14

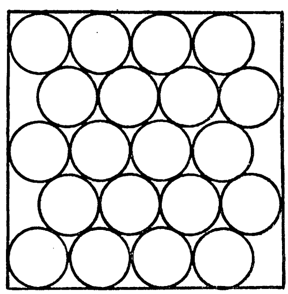

15

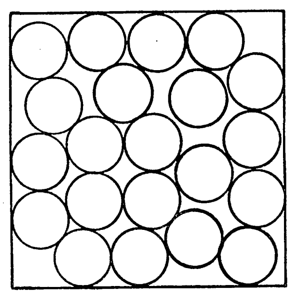

16

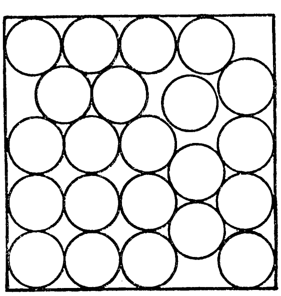

17

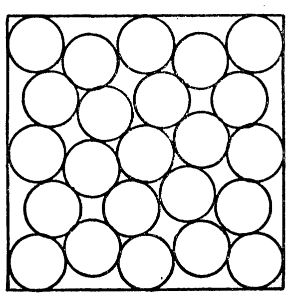

18

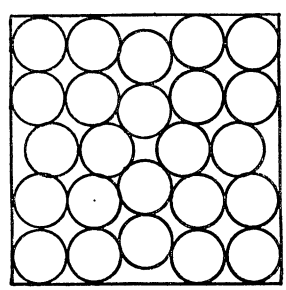

19

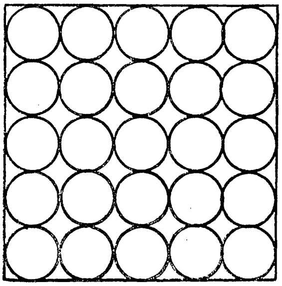

20


21

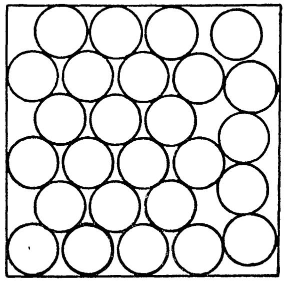

22


23

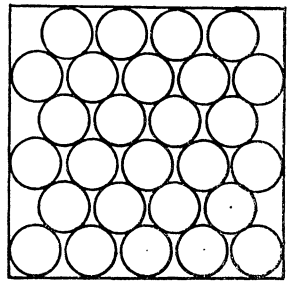

24



25



26

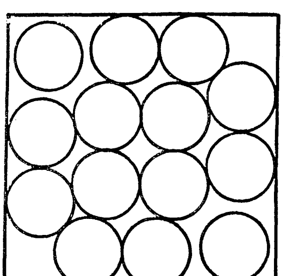


27

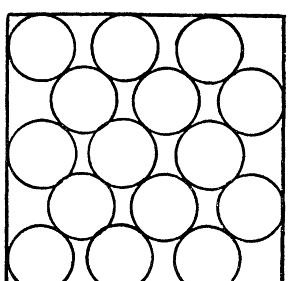
TABLE 2

n	Arrangement	m		$4d(n)/\pi$
14	Modified 16	$1/(1+\sqrt{3/2}+1/\sqrt{2})$	$= .3413$	.908
15	3,3,3,3,3	$\sqrt{41}/20$	$= .3202$	.882
17	21 circles minus 4	$1/\sqrt{2}-1/\sqrt{6}$	$= .2988$	.899
20	Irregular	$2-\sqrt{3}$	$= .2680$	.900
21	4,3,4,3,4,3	$\sqrt{61}/30$	$= .2603$	.896
23	Modified 25	$2/(4+\sqrt{6}+\sqrt{2})$	$= .2542$	.948
24	4,4,4,4,4,4	$\sqrt{74}/35$	$= .2458$	.933
24	Remove corner from 25	$4/(12+\sqrt{6}+\sqrt{2})$	$= .2522$	.974
30	5,5,5,5,5,5, shift	$(20-\sqrt{10})/75$	$= .2245$	1.008
39	6,5,6,5,6,5,6	$\sqrt{34}/30$	$= .1944$	1.033
42	6,6,6,6,6,6,6 shift	$(15-\sqrt{3})/72$	$= .1887$	1.017
52	7,6,7,6,7,6,7,6	$\sqrt{193}/84$	$= .1654$	1.047
80	8,8,8, $\cdots$	$\sqrt{34}/45$	$= .1296$	1.053
99	9,9,9, $\cdots$	$\sqrt{389}/170$	$= .1160$	1.070
120	10,10,10, $\cdots$	$\sqrt{482}/209$	$= .1050$	1.0844
143	11,11,11, $\cdots$, shift	$(40-\sqrt{5})/396$	$= .0954$	1.0839
161	12,11,12,11, $\cdots$	$\sqrt{653}/282$	$= .0906$	1.114
168	12,12,12, $\cdots$, shift	$\sqrt{17}/195$	$= .0871$	1.0804
188	13,12,13,12, $\cdots$	$\sqrt{193}/168$	$= .0827$	1.0967
270	15,15,15, $\cdots$	$\sqrt{1073}/476$	$= .0688$	1.1193
304	16,16,16, $\cdots$	$\sqrt{34}/90$	$= .0648$	1.1255
340	17,17,17, $\cdots$, shift	$(304-\sqrt{106}/4845$	$= .0606$	1.1107
∞	Regular dense packing			1.1547

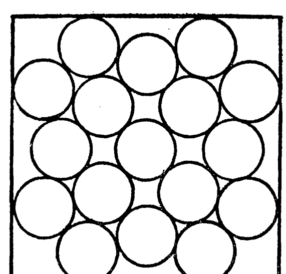
FIGURES FOR TABLE 2



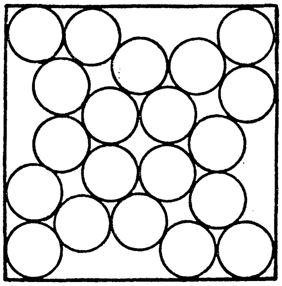
14



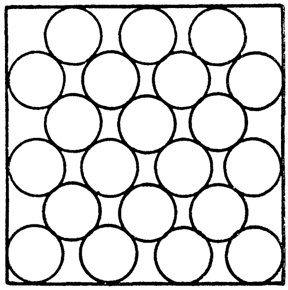
15



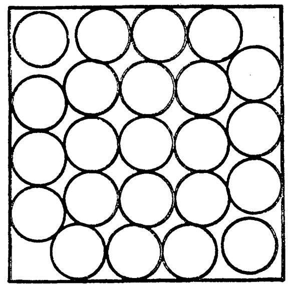
17



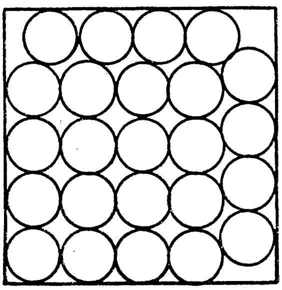
20



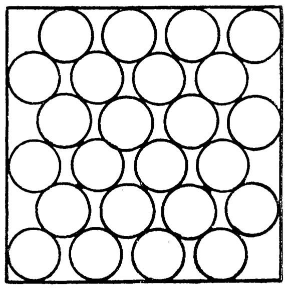
21



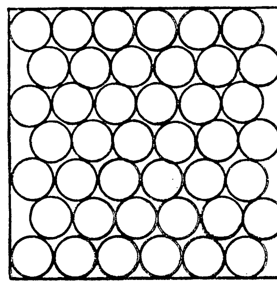
23



24



24



42

References

1. L. Fejes Toth, Lagerungen in der Ebene, auf der Kugel und im Raum, Springer, Berlin 1953.
2. M. Goldberg, Axially symmetric packing of equal circles on a sphere, Ann. Univ. Sci. Budapest. Sect. Math., 10 (1967) 37–48.
3. J. Schaer and A. Meir, On a geometric extremum problem, Canad. Math. Bull., 8 (1965) 21–27.
4. J. Schaer, The densest packing of 9 circles in a square, Canad. Math. Bull., 8 (1965) 273–277.
5. S. Kravitz, Packing cylinders into cylindrical containers, this MAGAZINE, 40 (1967) 65–71

INEQUALITIES FOR THE WALLIS PRODUCT

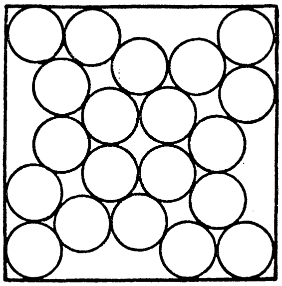
C. J. EVERETT, University of California, Los Alamos Scientific Laboratory

Introduction. In his entertaining book *Analytic Inequalities* [2, p. 47], N. D. Kazarinoff derives the inequalities

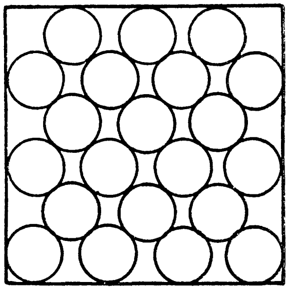
$$l_n \equiv 1/(1 + 1/2n) < W_n \equiv \{(2n-1)!!/(2n)!!\}^2 n\pi < 1 \tag{1}$$

for the "Wallis product" W_n in the following elementary way. Setting

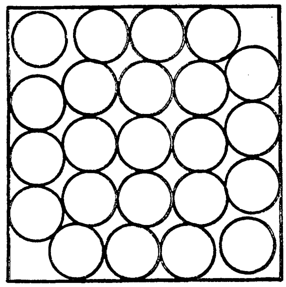
$$J_n = \int_0^{\pi/2} \sin^n \theta d\theta, n = 0, 1, 2, \cdots,$$



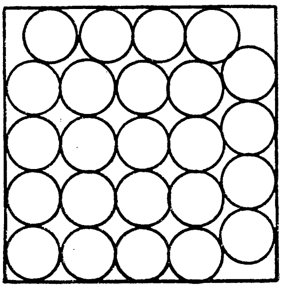
20



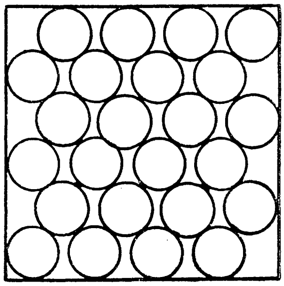
21



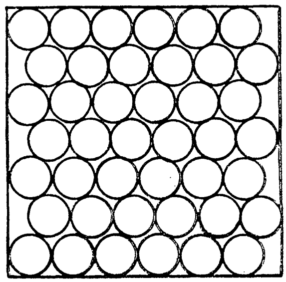
23



24



24



42

References

1. L. Fejes Toth, Lagerungen in der Ebene, auf der Kugel und im Raum, Springer, Berlin 1953.
2. M. Goldberg, Axially symmetric packing of equal circles on a sphere, Ann. Univ. Sci. Budapest. Sect. Math., 10 (1967) 37–48.
3. J. Schaer and A. Meir, On a geometric extremum problem, Canad. Math. Bull., 8 (1965) 21–27.
4. J. Schaer, The densest packing of 9 circles in a square, Canad. Math. Bull., 8 (1965) 273–277.
5. S. Kravitz, Packing cylinders into cylindrical containers, this MAGAZINE, 40 (1967) 65–71

INEQUALITIES FOR THE WALLIS PRODUCT

C. J. EVERETT, University of California, Los Alamos Scientific Laboratory

Introduction. In his entertaining book *Analytic Inequalities* [**2**, p. 47], N. D. Kazarinoff derives the inequalities

$$l_n \equiv 1/(1 + 1/2n) < W_n \equiv \{(2n-1)!!/(2n)!!\}^2 n\pi < 1 \tag{1}$$

for the "Wallis product" W_n in the following elementary way. Setting

$$J_n = \int_0^{\pi/2} \sin^n \theta d\theta, n = 0, 1, 2, \cdots,$$

where $J_0 = \pi/2$, $J_1 = 1$, integration by parts yields the recursion formula $J_{n+2} = (n+1)J_n/(n+2)$, whence

$$J_{2n} = \{(2n-1)!!/(2n)!!\}(\pi/2),\ J_{2n+1} = (2n)!!/(2n+1)!!.$$

Since $J_{n+1} < J_n$ by defintion, $J_{2n+1}J_{2n} < J_{2n}^2 < J_{2n-1}J_{2n}$, from which (1) follows upon substitution. The failure of induction to establish the inequality

$$W_n < u_n \equiv 1/(1 + 1/4n) \tag{2}$$

derived by D. K. Kazarinoff via the Γ-function, is also noted here (pp. 65, 66).

It is observed in Section 1 below that the very failure of induction in this case *implies* the truth of (2), and the trivial principle involved is exploited to secure a sharpening of both (1) and (2). In Section 2, a strong form of the log convexity property of the Γ-function is derived, and shown to be a generalized form of (2), thus providing an additional proof, more direct than that cited in [2]. A further sharpening of the bounds of Section 1 is also obtained from the Γ-function. While these results (except (1), (2)) and proofs are believed to be new, the primary aim of the present note is expository.

1. The bounds L_n and U_n. Since $u_n \to 1$, and $W_n \to 1$ by (1), we know $W_n/u_n \to 1$. Hence (2) would follow trivially if we also knew W_n/u_n were *increasing*. But this is just what appears in the failure of induction, since the final step of the desired chain $W_{n+1} = (W_{n+1}/W_n)W_n < (W_{n+1}/W_n)u_n < u_{n+1}$ is found to be *false*. (Success of such an induction would of course imply W_n/u_n decreasing, with W_1/u_1 its "most dangerous" term.) This situation suggests the following simple strategy.

Let $\{L_n\}$ and $\{U_n\}$ be sequences with limit 1, for which

$$L_{n+1}/L_n > W_{n+1}/W_n \equiv 1 + 1/(4n^2 + 4n) > U_{n+1}/U_n. \tag{3}$$

This is equivalent to saying that W_n/L_n is decreasing, and W_n/U_n is increasing, each to limit 1, from which it follows that

$$L_n < W_n < U_n. \tag{4}$$

To improve the bounds in (1), (2), we tentatively set

$$L_n = 1/\{1 + 1/(4n - a)\} < U_n = 1/\{1 + 1/(4n - b)\}$$

with $1 > a > b > 0$. It is readily verified that (3) then requires

$$4(1 - 2a)n - a(5 - a) < 0 < 4(1 - 2b)n - b(5 - b); \qquad n \geqq 1$$

indicating a choice of values $a = \frac{1}{2}$, $b < \rho \equiv \frac{1}{2}(13 - \sqrt{153}) \cong .315$. Clearly then, (4) will hold for

$$L_n = (8n - 1)/(8n + 1) > l_n \quad \text{and} \quad U_n = (4n - \rho)/(4n + 1 - \rho) < u_n.$$

2. Relation to the Γ-function. From the Weierstrass product

$$\Gamma(x) = (e^{-Cx}/x) \prod_{1}^{\infty} e^{x/i}/\{1 + (x/i)\}; \qquad x > 0$$

for the Γ-function [1, p. 16], one obtains, for positive x, a, the result

$$\Gamma(x)\Gamma(x+a)/\Gamma^2(x+\tfrac{1}{2}a) = \prod_0^\infty (x+i+\tfrac{1}{2}a)^2/(x+i)(x+i+a)$$

$$(5) \qquad = \prod_0^\infty \{1+(\tfrac{1}{4}a^2)/(x+i)(x+i+a)\}$$

$$(6) \qquad > 1+(\tfrac{1}{4}a^2)\sum_0^\infty 1/(x+i)(x+i+a).$$

Since the inequality $\Gamma(x)\Gamma(x+a)/\Gamma^2(x+\frac{1}{2}a)>1$ is equivalent to

$$\tfrac{1}{2}\{\log\Gamma(x)+\log\Gamma(x+a)\} > \log\Gamma(x+\tfrac{1}{2}a)$$

(6) may be regarded as a *strong* form of the "log convexity" property of the Γ-function [**1**, p. 13]. For integral $a=1, 2, \cdots$, the series in (6) is telescopic, and we may write instead [**3**, p. 234]

$$\Gamma(x)\Gamma(x+a)/\Gamma^2(x+\tfrac{1}{2}a)$$

$$(7) \qquad > 1+(\tfrac{1}{4}a)\sum_0^\infty \{1/(x+i)-1/(x+i+a)\}$$

$$= 1+(\tfrac{1}{4}a)\{1/x+1/x+1+\cdots+1/x+a-1\}.$$

In particular, for integral $x=n\geqq 1$, $a=1$, this becomes

$$(8) \qquad \Gamma(n)\Gamma(n+1)/\Gamma^2(n+\tfrac{1}{2}) > 1+1/4n.$$

Using the recursion formula $\Gamma(n+1)=n\Gamma(n)$ and the values $\Gamma(1)=1$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$ for the Γ-function [**1**, pp. 12, 13, 19], one finds that, in (8),

$$(9) \qquad \Gamma(n)\Gamma(n+1)/\Gamma^2(n+\tfrac{1}{2}) = 1/W_n$$

so that Kazarinoff's inequality (2) appears as a special instance of the strong log convexity property of $\Gamma(x)$ expressed in (6).

From (9), and its product form in (5), with $x=n$, $a=1$, we see that

$$1/W_n = \prod_0^\infty (1+x_{ni}); \qquad x_{ni}\equiv 1/4(n+i)(n+i+1).$$

We find by taking logarithms, and using the alternating character of the series for $\log(1+x_{ni})$, that this implies

$$1/4n = \sum_0^\infty x_{ni} > \log 1/W_n > \sum_0^\infty (x_{ni}-\tfrac{1}{2}x_{ni}^2)$$

$$> 1/4n-(1/96)\sum_0^\infty \{1/(n+i)^3-1/(n+i+1)^3\}$$

$$= 1/4n-1/96n^3.$$

Hence we have the following improvement on Section 1,

$$(10) \qquad L_n^* \equiv 1/\exp(1/4n) < W_n < U_n^* \equiv 1/\exp(1/4n-1/96n^3).$$

That $L_n^* > L_n$ follows from the relation $\log(1+\frac{1}{2}x)/(1-\frac{1}{2}x) > x$ for $x = 1/4n$. To verify $U_n^* < U_n (n \geqq 2)$ is more troublesome, and we omit the proof.

Author's work performed under the auspices of the U. S. Atomic Energy Commission.

References

1. E. Artin, The Gamma Function, Holt, Rinehart and Winston, New York, 1964.
2. N. D. Kazarinoff, Analytic Inequalities, Holt, Rinehart and Winston, New York, 1961.
3. K. Knopp, Theory and Application of Infinite Series, 2nd Eng. ed., Hafner, New York, 1947.

GEOMETRY OF GENERALIZED INVERSES

A. R. AMIR-MOEZ and T. G. NEWMAN, Texas Tech University

This expository note intends to explain a geometric interpretation of generalized inverses of linear transformations. We shall use standard notation of vector spaces. To make the ideas as simple as possible the examples given will be in two dimensions.

1. Introduction to generalized inverses. Throughout this paper V will denote a finite dimensional vector space over a field F and A will denote a linear transformation on V. Let the range of A be R and the null space of A be N. Vectors will be denoted by small Greek letters.

The simplest way to introduce generalized inverses is by means of linear equations of the form

$$A\xi = \delta \tag{1}$$

where we wish to solve for ξ. It is natural to try to solve the above equation by means of a linear transformation applied to δ. In other words, we seek a linear transformation B on V with the property that $\xi = B\delta$ is a solution of equation (1) whenever a solution exists; that is, whenever δ is in R. Such a linear transformation always exists, as we shall later see, but is generally not unique, and will be referred to as a generalized inverse of A. It can be shown without difficulty that the condition $ABA = A$ is necessary and sufficient for B to be a generalized inverse of A, although this will not be important in what follows.

2. A special case. Let us restrict our attention to the case in which $R \cap N = (0)$. For example, when A is a normal matrix over the real number field.

A generalized inverse of A can be constructed by first restricting A to its range. The resulting linear transformation on R is then nonsingular and has an inverse $(A \mid R)^{-1}$, also on R. Now let P be the projection onto R along the complementary space N. Then we define

$$B = (A \mid R)^{-1}P. \tag{2}$$

Now consider the equation $A\xi = \delta$ and let $\xi_0 = B\delta$. Observing that $A(A \mid R)^{-1}$ is the identity on R, we have

$$A\xi_0 = A(A \mid R)^{-1}P\delta = P\delta.$$

That $L_n^* > L_n$ follows from the relation $\log(1+\frac{1}{2}x)/(1-\frac{1}{2}x) > x$ for $x = 1/4n$. To verify $U_n^* < U_n (n \geqq 2)$ is more troublesome, and we omit the proof.

Author's work performed under the auspices of the U. S. Atomic Energy Commission.

References

1. E. Artin, The Gamma Function, Holt, Rinehart and Winston, New York, 1964.
2. N. D. Kazarinoff, Analytic Inequalities, Holt, Rinehart and Winston, New York, 1961.
3. K. Knopp, Theory and Application of Infinite Series, 2nd Eng. ed., Hafner, New York, 1947.

GEOMETRY OF GENERALIZED INVERSES

A. R. AMIR-MOEZ and T. G. NEWMAN, Texas Tech University

This expository note intends to explain a geometric interpretation of generalized inverses of linear transformations. We shall use standard notation of vector spaces. To make the ideas as simple as possible the examples given will be in two dimensions.

1. Introduction to generalized inverses. Throughout this paper V will denote a finite dimensional vector space over a field F and A will denote a linear transformation on V. Let the range of A be R and the null space of A be N. Vectors will be denoted by small Greek letters.

The simplest way to introduce generalized inverses is by means of linear equations of the form

$$A\xi = \delta \tag{1}$$

where we wish to solve for ξ. It is natural to try to solve the above equation by means of a linear transformation applied to δ. In other words, we seek a linear transformation B on V with the property that $\xi = B\delta$ is a solution of equation (1) whenever a solution exists; that is, whenever δ is in R. Such a linear transformation always exists, as we shall later see, but is generally not unique, and will be referred to as a generalized inverse of A. It can be shown without difficulty that the condition $ABA = A$ is necessary and sufficient for B to be a generalized inverse of A, although this will not be important in what follows.

2. A special case. Let us restrict our attention to the case in which $R \cap N = (0)$. For example, when A is a normal matrix over the real number field.

A generalized inverse of A can be constructed by first restricting A to its range. The resulting linear transformation on R is then nonsingular and has an inverse $(A \mid R)^{-1}$, also on R. Now let P be the projection onto R along the complementary space N. Then we define

$$B = (A \mid R)^{-1}P. \tag{2}$$

Now consider the equation $A\xi = \delta$ and let $\xi_0 = B\delta$. Observing that $A(A \mid R)^{-1}$ is the identity on R, we have

$$A\xi_0 = A(A \mid R)^{-1}P\delta = P\delta.$$

Therefore, $A\xi_0=\delta$ if and only if $P\delta=\delta$, which is to say δ is in R. It follows that equation (2) defines a generalized inverse of A.

Example 1. Consider the 2×2 matrix

$$A=\begin{pmatrix}1 & -3\\ 2 & -6\end{pmatrix}.$$

Without loss of generality we assume the basis is orthonormal in the Euclidean plane. R has the equation $y=2x$ and N has the equation $3y=x$ (Figure 1).

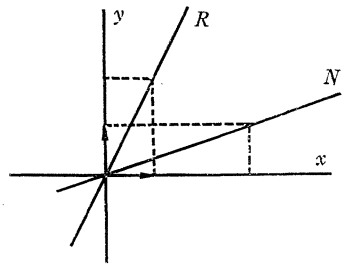


FIG. 1.

Here

$$P=-\frac{1}{5}\begin{pmatrix}1 & -3\\ 2 & -6\end{pmatrix}.$$

Note that for ξ in R we have $A\xi=-5\xi$. Thus $(A\,|\,R)^{-1}\xi=-\frac{1}{5}\xi$. Therefore

$$B=(A\,|\,R)^{-1}P=-\frac{1}{5}P=\frac{1}{25}\begin{pmatrix}1 & -3\\ 2 & -6\end{pmatrix}.$$

Now to solve the equation $A\xi=\delta$ it will suffice to try $\xi_0=B\delta$. One easily verifies that ξ_0 is a solution if and only if δ is in R. Thus B is a generalized inverse of A.

3. The general case. The assumption $R\cap N=(0)$ above was essential only in one respect. Namely R and N were complementary subspaces. To define a generalized inverse without making the above assumption we need only provide complementary subspaces for R and N. Let U be a subspace complementary to N and let W be complementary to R. From elementary linear algebra we know that $A\,|\,U$ is a nonsingular mapping of U onto R and has an inverse mapping R onto U. Let P be the projection onto R along W. We define B as in equation (2) by

$$B=(A\,|\,U)^{-1}P. \tag{3}$$

Now consider the equation $A\xi=\delta$. Letting $\xi_0=B\delta$ and proceeding exactly as before we have $A\xi_0=P\delta$. Thus, $A\xi_0=\delta$ if and only if δ is in R. Hence B is a generalized inverse of A.

It is worth mentioning at this point that $Q=AB$ is the projection onto U along N and that $P=BA$.

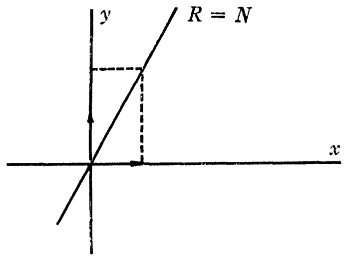


FIG. 2.

Example 2. Consider the 2×2 matrix

$$A = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}.$$

It is clear that R and N both have the equation $y=2x$ (Figure 2). Let us define a complement U of N by the equation $y=-x$ and define a complement W of R by $y=x$. We easily see that the projection P on R along W is given by

$$P = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}.$$

Also,

$$(A \mid U)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore

$$(A \mid U)^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A direct calculation gives

$$(A \mid U)^{-1}P\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x-y \\ y-x \end{pmatrix}.$$

Hence

$$B = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

4. The Moore-Penrose generalized inverse. The Moore-Penrose generalized inverse of A can be defined as the unique linear transformation A^+ satisfying the equations

(a) $AA^+A = A$ (c) $(AA^+)^* = AA^+$

(b) $A^+AA^+ = A^+$ (d) $(A^+A)^* = A^+A$

where T^* is the adjoint of T relative to a given inner product. In the notation of

the preceding section equations (c) and (d) simply require that $P=AA^+$ and $Q=A^+A$ be orthogonal projections. One easily observes that if in Section 3 we choose U to be the orthogonal complement of N and W to be the orthogonal complement of R, then we obtain the Moore-Penrose generalized inverse of A.

References

1. J. S. Frame, Matrix functions and applications, IEEE Spectrum, (1964) 209–220.
2. E. H. Moore, General analysis, Part I, Mem. Amer. Philos. Soc., I (1935).
3. R. A. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc., 51 (1955) 406–413.
4. D. W. Robinson, On the generalized inverse of an arbitrary linear transformation, Amer. Math. Monthly, 69 (1962) 412–416.

ON THE NUMBER OF SUBSETS OF A FINITE SET

DAVID S. GREENSTEIN, Northeastern Illinois State College

In Freshman courses in "Modern Mathematics", it is easy to find students who have difficulty grasping the combinatorial reasoning usually employed to show that an n-element set has 2^n subsets. I have found the following way of showing why adding a new element to a set doubles the number of subsets. This approach plus the obvious fact that the empty set has exactly one subset makes it easy to find the number of subsets of an n-element set in n easy steps.

Let A be a finite set and let $b\notin A$. Then each subset S of A "gives birth" to the two subsets S and $S\cup\{b\}$ of $A\cup\{b\}$. Furthermore, each subset of $A\cup\{b\}$ has a unique "birth" from some subset of A. Thus adding b has doubled the number of subsets.

ANSWERS

A468. Taking the cube root of both sides and dividing by 3 gives an obvious proof since the average of three positive numbers is not smaller than their geometric mean.

A469. $(x-[x])x=[x]^2$ or $x^2-[x]x-[x]^2=0$ and $x=[x](1+\sqrt{5})/2$.

Then $[x]+1\geqq[x](1+\sqrt{5})/2$ and $[x]\leqq 2/(\sqrt{5}-1)<2$. Thus $[x]=0$ or 1 and $x=0$ or $(1+\sqrt{5})/2$.

A470. $2^{5n+1}+5^{n+2}=2(27+5)^n+25+5^n=2k27+(2+25)5^n$.

A471. Without loss of generality, it may be assumed that k is the least integer such that $x^2+y^2=3^k$. Then $x^2+y^2\equiv 0 \pmod 3$ from which it follows that $x\equiv y\equiv 0 \pmod 3$. Therefore $x=3m$ and $y=3n$ which yields $(3m)^2+(3n)^2=3^k$ or $m^2+n^2=3^{k-2}$, a contradiction of the minimal condition on k. The conclusion follows.

(Quickies on page 56.)

the preceding section equations (c) and (d) simply require that $P=AA^+$ and $Q=A^+A$ be orthogonal projections. One easily observes that if in Section 3 we choose U to be the orthogonal complement of N and W to be the orthogonal complement of R, then we obtain the Moore-Penrose generalized inverse of A.

References

1. J. S. Frame, Matrix functions and applications, IEEE Spectrum, (1964) 209–220.
2. E. H. Moore, General analysis, Part I, Mem. Amer. Philos. Soc., I (1935).
3. R. A. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc., 51 (1955) 406–413.
4. D. W. Robinson, On the generalized inverse of an arbitrary linear transformation, Amer. Math. Monthly, 69 (1962) 412–416.

ON THE NUMBER OF SUBSETS OF A FINITE SET

DAVID S. GREENSTEIN, Northeastern Illinois State College

In Freshman courses in "Modern Mathematics", it is easy to find students who have difficulty grasping the combinatorial reasoning usually employed to show that an n-element set has 2^n subsets. I have found the following way of showing why adding a new element to a set doubles the number of subsets. This approach plus the obvious fact that the empty set has exactly one subset makes it easy to find the number of subsets of an n-element set in n easy steps.

Let A be a finite set and let $b \notin A$. Then each subset S of A "gives birth" to the two subsets S and $S \cup \{b\}$ of $A \cup \{b\}$. Furthermore, each subset of $A \cup \{b\}$ has a unique "birth" from some subset of A. Thus adding b has doubled the number of subsets.

ANSWERS

A468. Taking the cube root of both sides and dividing by 3 gives an obvious proof since the average of three positive numbers is not smaller than their geometric mean.

A469. $(x-[x])x=[x]^2$ or $x^2-[x]x-[x]^2=0$ and $x=[x](1+\sqrt{5})/2$.

Then $[x]+1 \geqq [x](1+\sqrt{5})/2$ and $[x] \leqq 2/(\sqrt{5}-1)<2$. Thus $[x]=0$ or 1 and $x=0$ or $(1+\sqrt{5})/2$.

A470. $2^{5n+1}+5^{n+2}=2(27+5)^n+25+5^n=2k27+(2+25)5^n$.

A471. Without loss of generality, it may be assumed that k is the least integer such that $x^2+y^2=3^k$. Then $x^2+y^2\equiv 0 \pmod 3$ from which it follows that $x\equiv y\equiv 0 \pmod 3$. Therefore $x=3m$ and $y=3n$ which yields $(3m)^2+(3n)^2=3^k$ or $m^2+n^2=3^{k-2}$, a contradiction of the minimal condition on k. The conclusion follows.

(Quickies on page 56.)

CONGRUENCE-PRESERVING MAPPINGS

CARL G. TOWNSEND, Southern Illinois University

We consider the Euclidean plane P with the usual distance between points p and q denoted by $d(p, q)$. To begin with, we have the following definition:

DEFINITION 1.—*Two subsets A and B of P are congruent if there exists a one-to-one function f mapping A onto B such that $d(p, q) = d(f(p), f(q))$ for all p and q in A.*

This note will be concerned with a special class of such functions.

DEFINITION 2.—*A function f mapping the plane P into itself is a congruence-preserving function if A is congruent to $f(A)$ for all subsets A of P.*

Congruence-preserving functions are closely related to the following types of functions.

DEFINITION 3.—*A function f mapping the plane P into itself is distance-preserving if $d(p, q) = d(f(p), f(q))$ for all points p and q in P.*

DEFINITION 4.—*A function f mapping the plane P into itself is unit distance-preserving if $d(f(p), f(q)) = 1$ whenever $d(p, q) = 1$.*

THEOREM 1.—*Let f be a function from P into P. If f is distance-preserving then f is congruence-preserving. If f is congruence-preserving then f is unit distance-preserving.*

The proof of Theorem 1 is nothing more than a close look at the various definitions involved. A more surprising and less trivial statement is found in the following theorem.

THEOREM 2.—*Let f be a one-to-one unit distance-preserving function from the plane into itself. Then f is distance-preserving.*

Proof.—Let f be a one-to-one unit distance-preserving function. It is easy to show that f preserves all integral distances.

We now show that f preserves all rational distances. Let m and n be positive integers and p and q be two points such that $d(p, q) = m/n$. Assume that $n > m$, and consider the following construction.

Extend the segment $\overline{pq}$ to the point r such that $d(p, r) = m$. Choose s and s' such that $d(p, s) = d(r, s) = d(p, s') = d(r, s') = n$ where $s \neq s'$. Choose t and t' on the segments $\overline{ps}$ and $\overline{ps'}$ respectively such that $d(p, t) = d(p, t') = 1$.

Then $d(p, r)/d(p, q) = m/(m/n) = n/1 = d(p, s)/d(p, t)$. Therefore the triangles ptq and psr are similar and hence, $d(t, q) = 1$. Likewise $d(t', q) = 1$. It now follows that $d(f(p), f(q)) = m/n$.

Since every rational number lies between two consecutive integers we have f rational distance-preserving.

The next step is to show that f is continuous. Let $p \in P$ and let $\epsilon > 0$ be given. Choose any $\delta > 0$ such that $\delta < \epsilon/2$ and suppose $d(p, q) < \delta$ where $q \neq p$.

There exists a point r on the perpendicular bisector of the segment $\overline{pq}$ such that $d(p, r)$ is rational and $d(p, r) < \delta$. Then:

$$d(f(p), f(q)) \leqq d(f(p), f(r)) + d(f(r), f(q)) = d(p, r) + d(r, q) < \delta + \delta < \epsilon.$$

Hence, f is continuous.

Since composites of continuous functions are continuous, it follows that $d(p, q) = d(f(p), f(q))$ for all p and q in P.

Remarks. The problem of showing that a rational distance-preserving function is distance-preserving is found in [1]. One question that remains to be answered is on the one-to-one condition in Theorem 2, i.e., is it true that unit distance-preserving functions are one-to-one? Another question which arises is that of extending the results to other spaces. In [1] it is shown that 1-1 rational distance-preserving functions of the line into itself are not necessarily distance-preserving. (Let $f(x) = x$ if x is rational and let $f(x) = x+1$ if x is irrational.)

Reference

1. L. E. Bush, The William Lowell Putnam Mathematical Competition, Amer. Math. Monthly., 64 (1957) 21.

MORE ON VECTOR REPRESENTATION OF RIGID BODY ROTATION

HARRY W. HICKEY, Arlington, Va.

Wong [1] recently presented in this MAGAZINE two new derivations for the formula for the vector r produced by rotating the vector r_0 through an angle θ about an axis represented by unit vector u. These derivations appear to be simpler than the usual ones in the literature. This note will present a derivation simpler still, in that it makes no use of calculus, but follows a simple, intuitive geometrical argument.

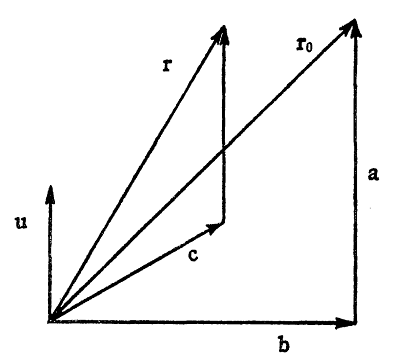


FIG. 1.

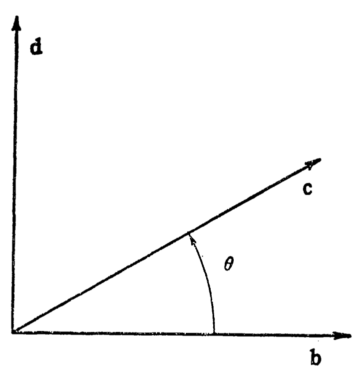


FIG. 2.

There exists a point r on the perpendicular bisector of the segment $\overline{pq}$ such that $d(p, r)$ is rational and $d(p, r) < \delta$. Then:

$$d(f(p), f(q)) \leqq d(f(p), f(r)) + d(f(r), f(q)) = d(p, r) + d(r, q) < \delta + \delta < \epsilon.$$

Hence, f is continuous.

Since composites of continuous functions are continuous, it follows that $d(p, q) = d(f(p), f(q))$ for all p and q in P.

Remarks. The problem of showing that a rational distance-preserving function is distance-preserving is found in [1]. One question that remains to be answered is on the one-to-one condition in Theorem 2, i.e., is it true that unit distance-preserving functions are one-to-one? Another question which arises is that of extending the results to other spaces. In [1] it is shown that 1-1 rational distance-preserving functions of the line into itself are not necessarily distance-preserving. (Let $f(x) = x$ if x is rational and let $f(x) = x+1$ if x is irrational.)

Reference

1. L. E. Bush, The William Lowell Putnam Mathematical Competition, Amer. Math. Monthly., 64 (1957) 21.

MORE ON VECTOR REPRESENTATION OF RIGID BODY ROTATION

HARRY W. HICKEY, Arlington, Va.

Wong [1] recently presented in this MAGAZINE two new derivations for the formula for the vector r produced by rotating the vector r_0 through an angle θ about an axis represented by unit vector u. These derivations appear to be simpler than the usual ones in the literature. This note will present a derivation simpler still, in that it makes no use of calculus, but follows a simple, intuitive geometrical argument.

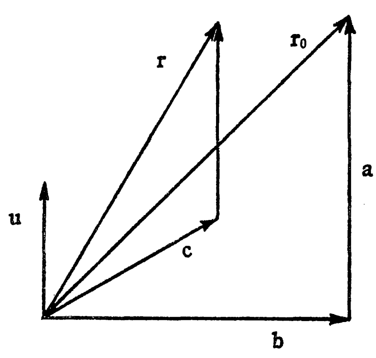


FIG. 1.

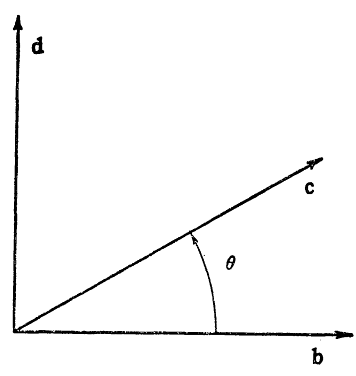


FIG. 2.

In the first of our two figures, we present vectors $\boldsymbol{r}$, $\boldsymbol{r}_0$, and $\boldsymbol{u}$, viewed obliquely to P, a plane normal to $\boldsymbol{u}$. This figure also shows vectors $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{a}$, being respectively the component of $\boldsymbol{r}_0$ normal to $\boldsymbol{u}$, the component of $\boldsymbol{r}$ normal to $\boldsymbol{u}$, and the component (common to $\boldsymbol{r}_0$ and $\boldsymbol{r}$) parallel to $\boldsymbol{u}$. Clearly $\boldsymbol{a}=(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}$, while $\boldsymbol{b}=\boldsymbol{r}_0-\boldsymbol{a}=\boldsymbol{r}_0-(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}$. Also $\boldsymbol{r}=\boldsymbol{b}+\boldsymbol{c}$, so that, having the formula for $\boldsymbol{b}$, we need only find $\boldsymbol{c}$. To do this, refer to the second figure, drawn in the plane of P. Here we have constructed vector $\boldsymbol{d}$, of the same length as b but normal to $\boldsymbol{b}$ and so oriented that the $90°$ angle from $\boldsymbol{b}$ to $\boldsymbol{d}$ is in the positive sense. That is, $\boldsymbol{d}=\boldsymbol{u}\times\boldsymbol{b}$. Then $\boldsymbol{c}=\boldsymbol{b}\cos\theta+\boldsymbol{d}\sin\theta$, whence:

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{a}+\boldsymbol{c} = \boldsymbol{a}+\boldsymbol{b}\cos\theta+\boldsymbol{d}\sin\theta \\ &= (\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}+[\boldsymbol{r}_0-(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}]\cos\theta+\boldsymbol{u}\times[\boldsymbol{r}_0-(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}]\sin\theta.\end{aligned}$$

The last term simplifies a little because the cross-product $\boldsymbol{u}\times\boldsymbol{u}$ vanishes, and we have:

$$\begin{aligned}\boldsymbol{r} &= (\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}+[\boldsymbol{r}_0-(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}]\cos\theta+(\boldsymbol{u}\times\boldsymbol{r}_0)\sin\theta \\ &= (1-\cos\theta)\cdot(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}+\cos\theta\cdot\boldsymbol{r}_0+\sin\theta\cdot(\boldsymbol{u}\times\boldsymbol{r}_0)\end{aligned}$$

—the same as Wong's result.

Reference

1. J. S. Wong, On vector representation of rigid body rotation, this MAGAZINE, 41 (1968) 28–29.

OPEN MAPPINGS AND THE FUNDAMENTAL THEOREM OF ALGEBRA

R. L. THOMPSON, SUNY at Binghamton

It is well known that a nonconstant analytic function is an open mapping. The usual proof of this result is based upon the complex integral together with the power series expandibility of the function. Since openness is a topological property, it was natural for mathematicians to seek an elementary proof which uses a minimum of analytic machinery. Eggleston and Ursell [1] as well as Titus and Young [2] have done this. Although their proofs are elementary (in the above sense) they are too sophisticated for an undergraduate complex variables course. It is possible, however, to present an elementary and unsophisticated proof for the openness of a nonconstant complex polynomial. It is shown here that this result is in fact equivalent to the Fundamental Theorem of Algebra and thus a simple proof of the latter yields a simple proof that every nonconstant polynomial is an open mapping.

THEOREM 1. *The following two statements are equivalent:*
(i) *The Fundamental Theorem of Algebra.*
(ii) *Every nonconstant complex polynomial is an open mapping.*

In the first of our two figures, we present vectors $\boldsymbol{r}$, $\boldsymbol{r}_0$, and $\boldsymbol{u}$, viewed obliquely to P, a plane normal to $\boldsymbol{u}$. This figure also shows vectors $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{a}$, being respectively the component of $\boldsymbol{r}_0$ normal to $\boldsymbol{u}$, the component of $\boldsymbol{r}$ normal to $\boldsymbol{u}$, and the component (common to $\boldsymbol{r}_0$ and $\boldsymbol{r}$) parallel to $\boldsymbol{u}$. Clearly $\boldsymbol{a}=(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}$, while $\boldsymbol{b}=\boldsymbol{r}_0-\boldsymbol{a}=\boldsymbol{r}_0-(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}$. Also $\boldsymbol{r}=\boldsymbol{b}+\boldsymbol{c}$, so that, having the formula for $\boldsymbol{b}$, we need only find $\boldsymbol{c}$. To do this, refer to the second figure, drawn in the plane of P. Here we have constructed vector $\boldsymbol{d}$, of the same length as b but normal to $\boldsymbol{b}$ and so oriented that the 90° angle from $\boldsymbol{b}$ to $\boldsymbol{d}$ is in the positive sense. That is, $\boldsymbol{d}=\boldsymbol{u}\times\boldsymbol{b}$. Then $\boldsymbol{c}=\boldsymbol{b}\cos\theta+\boldsymbol{d}\sin\theta$, whence:

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{a}+\boldsymbol{c} = \boldsymbol{a}+\boldsymbol{b}\cos\theta+\boldsymbol{d}\sin\theta\\ &= (\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}+[\boldsymbol{r}_0-(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}]\cos\theta+\boldsymbol{u}\times[\boldsymbol{r}_0-(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}]\sin\theta.\end{aligned}$$

The last term simplifies a little because the cross-product $\boldsymbol{u}\times\boldsymbol{u}$ vanishes, and we have:

$$\begin{aligned}\boldsymbol{r} &= (\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}+[\boldsymbol{r}_0-(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}]\cos\theta+(\boldsymbol{u}\times\boldsymbol{r}_0)\sin\theta\\ &= (1-\cos\theta)\cdot(\boldsymbol{u}\cdot\boldsymbol{r}_0)\boldsymbol{u}+\cos\theta\cdot\boldsymbol{r}_0+\sin\theta\cdot(\boldsymbol{u}\times\boldsymbol{r}_0)\end{aligned}$$

—the same as Wong's result.

Reference

1. J. S. Wong, On vector representation of rigid body rotation, this MAGAZINE, 41 (1968) 28–29.

OPEN MAPPINGS AND THE FUNDAMENTAL THEOREM OF ALGEBRA

R. L. THOMPSON, SUNY at Binghamton

It is well known that a nonconstant analytic function is an open mapping. The usual proof of this result is based upon the complex integral together with the power series expandibility of the function. Since openness is a topological property, it was natural for mathematicians to seek an elementary proof which uses a minimum of analytic machinery. Eggleston and Ursell [1] as well as Titus and Young [2] have done this. Although their proofs are elementary (in the above sense) they are too sophisticated for an undergraduate complex variables course. It is possible, however, to present an elementary and unsophisticated proof for the openness of a nonconstant complex polynomial. It is shown here that this result is in fact equivalent to the Fundamental Theorem of Algebra and thus a simple proof of the latter yields a simple proof that every nonconstant polynomial is an open mapping.

THEOREM 1. *The following two statements are equivalent*:
(i) *The Fundamental Theorem of Algebra.*
(ii) *Every nonconstant complex polynomial is an open mapping.*

Proof. (i) implies (ii). Suppose $f(z)=z^m+a_{m-1}z^{m-1}+\cdots+a_0$ is not open. Then there is an open set U and a point $p\in U$ such that $f(p)\notin \operatorname{int} f(U)$. This implies that there is an $r>0$ such that $f(p)\notin \operatorname{int} f(S_r(p))$. We can thus pick b such that $\frac{1}{2}r^m>|f(p)-b|$, but $b\notin f(S_r(p))$. Since each complex polynomial has a zero, there exist $z_1,\cdots,z_m$ such that $f(z)-b=(z-z_1)(z-z_2)\cdots(z-z_m)$. Since $b\notin f(S_r(p))$, it follows that $z_j\notin S_r(p)$ for $j=1,\cdots,m$. Therefore,

$$|f(p)-b| = |p-z_1||p-z_2|\cdots|p-z_m| \geqq r^m.$$

This contradiction implies that $f(z)$ is open.

(ii) implies (i). Suppose that $f(z)$ is open and has no zero. If $z\neq 0$, $f(z)$ can be written as $f(z)=z^m[1+a_{m-1}/z+\cdots+a_0/z^m]$. Hence, there must exist an $R>0$ such that $|f(z)|>\frac{1}{2}|z^m|>|a_0|$ if $|z|>R$. The complex plane C is the union of $A=\{z:|z|\leqq R\}$ and $B=\{z:|z|>R\}$. We have $a_0\in f(A)$, while $|f(z)|>|a_0|$ for every $z\in B$. Since $f(C)$ is the union of $f(A)$ and $f(B)$, this means that the distance from 0 to $f(C)$ is equal to the distance from 0 to the compact set $f(A)$. Thus there is a point $v\in A$ such that this distance is equal to $|f(v)|$. Since $f(v)\neq 0$, this implies that there are no points of $f(C)$ on the line segment from $f(v)$ to 0. This contradicts the assumption that $f(C)$ is open. Therefore, $f(v)=0$.

Starting with the v obtained in Theorem 1, an elementary proof of the Fundamental Theorem of Algebra can easily be completed without using the condition that $f(z)$ is open.

THEOREM 2. *The nonconstant complex polynomial*

$$f(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0 \textit{ has a zero.}$$

Proof. Suppose that $f(v)\neq 0$, where v is as in Theorem 1. Since $f(z+v)$ is a polynomial in z, it has a term of minimum positive degree j with a coefficient of c. Let $b=(-q/c)^{1/j}$, where $q=f(v)$. Then $f(rb+v)$ can be written as $f(rb+v)=q-qr^j+qr^kg(r)$, where $k>j$ and $g(r)$ is a polynomial in r. Let M be an upper bound of $|g(r)|$ for $0<r<1$. If the real number r is now chosen so that $r^{k-j}<1/2M$ and $0<r<1$, then

$$\begin{aligned} |f(rb+v)| &= |q||(1-r^j+r^kg(r))| \\ &< |q||(1-r^j)| + |q|r^kM < |q|(1-r^j)+\tfrac{1}{2}|q|r^j < |q|. \end{aligned}$$

This contradicts the fact that $|q|=|f(v)|$ is the distance from 0 to $f(C)$. Hence v is a zero of $f(z)$.

References

1. H. C. Eggleston and H. D. Ursell, On the lightness and strong interiority of analytic functions, J. London Math. Soc., 27 (1952) 260–271.

2. C. J. Titus and G. S. Young, A Jacobian condition for interiority, Michigan Math. J., 1 (1952) 89–94.

BINARY DIGITAL ARITHMETIC

WILLIAM J. PERVIN, Drexel Institute of Technology

In this note we will consider an elementary way to introduce the polynomial ring $\mathbf{Z}_2[x]$ as an example of an integral domain before studying polynomial rings *per se*. This will lead to some interesting and challenging problems for the student.

Each positive integer a may be expressed in a unique binary notation $(a_k \cdots a_1a_0)_2$ where

$$a = 2^k a_k + \cdots + 2^2 a_2 + 2a_1 + a_0$$

with each $a_i \in \mathbf{Z}_2$ and $a_k \neq 0$ (k may be called the degree of a). Zero may be written $(0)_2$. The sum of two nonnegative integers $a = (a_k \cdots a_1a_0)_2$ and $b = (b_j \cdots b_1b_0)_2$, written $a \oplus b$, is defined to be the nonnegative integer $c = (c_n \cdots c_1c_0)_2$ where $c_i = a_i + b_i \pmod 2$, letting $a_i = 0$ for $i > k$ and $b_i = 0$ for $i > j$. For example,

$$3 \oplus 7 = (11)_2 \oplus (111)_2 = (100)_2 = 4.$$

The mechanics of this addition are quite simple and can be put in the usual form:

$$\begin{array}{rl} (110101)_2 = & 53 \\ (101101)_2 = & 45 \\ \hline (11000)_2 = & 24 \end{array} \qquad \begin{array}{rl} (100101)_2 = & 37 \\ (10100)_2 = & 20 \\ \hline (110001)_2 = & 49. \end{array}$$

The solutions to many NIM-type games may be found by using these "binary digital" sums (see [1] p. 24).

The product $a \times b$ of $a = (a_k \cdots a_1a_0)_2$ and $b = (b_j \cdots b_1b_0)_2$ is defined to be $(c_n \cdots c_1c_0)_2$ where

$$c_i = \sum_{r=0}^{i} a_r b_{i-r} \qquad (\mathrm{mod}\ 2)$$

letting $a_i = 0$ for $i > k$ and $b_i = 0$ for $i > j$. The rule is to multiply as usual but then use the digital addition; for example:

$$\begin{array}{rl} (1011)_2 = & 11 \\ (111)_2 = & 7 \\ \hline 1011 & \\ 1011 & \\ 1011 & \\ \hline (110001)_2 = & 49 \end{array} \qquad \begin{array}{rl} (1101)_2 = & 13 \\ (101)_2 = & 5 \\ \hline 1101 & \\ 0 & \\ 1101 & \\ \hline (111001)_2 = & 57 \end{array} \qquad \begin{array}{rl} (11)_2 = & 3 \\ (11)_2 = & 3 \\ \hline 11 & \\ 11 & \\ \hline (101)_2 = & 5. \end{array}$$

One may now prove directly that this yields an integral domain, or Euclidean domain (see [3] p. 104), which is not a field.

With some work a student can obtain a short list of primes in this system such as: 2, 3, 7, 11, 13, 19, 25, 31, 37, 41, 47, 55, 59, 61, $\cdots$, (see [5] or [6] for longer lists). It is clear that the prime number 7, for example, corresponds to the irreducible polynomial x^2+x+1 over $\mathbf{Z}_2$. With such a list available, we may consider the quotient fields modulo the ideal generated by a prime in some detail. In general the quotient field will have a cyclic multiplication group of nonzero elements (see [3] p. 317) and the additive group will be the product of as many copies of $\mathbf{Z}_2$ as the degree of the prime. Thus we may obtain the Galois fields $GF[2^n]$ in this manner.

Since this domain is of characteristic two ($\mathbf{x}\oplus\mathbf{x}=\mathbf{0}$ for all $\mathbf{x}$), it follows that the equation $\mathbf{x}^n\oplus\mathbf{y}^n=\mathbf{z}^n$ has solutions in positive integers (under digital arithmetic) whenever n is a power of two; furthermore, given any pair $\mathbf{x}$, $\mathbf{y}$ the solution is uniquely given by $\mathbf{z}=\mathbf{x}\oplus\mathbf{y}$ (see [2] Problem 12, p. 438). While Fermat's Last Theorem does not hold in this domain, it is interesting to ask if perhaps there are no solutions when n is not a power of two. We note that $\mathbf{x}^3\oplus\mathbf{y}^3=(\mathbf{x}\oplus\mathbf{y})\times(\mathbf{x}^2\oplus\mathbf{x}\mathbf{y}\oplus\mathbf{y}^2)$, but no solutions have been found.

The preparation of this paper was supported in part by the National Science Foundation under GP-4432 to the University of Wisconsin.

References

1. C. Berge, The Theory of Graphs, Wiley, New York, 1962.
2. W. E. Deskins, Abstract Algebra, Macmillan, New York, 1964.
3. I. N. Herstein, Topics in Algebra, Blaisdell, New York, 1964.
4. D. J. Lewis, Introduction to Algebra, Harper and Row, New York, 1965.
5. R. C. Tausworthe, Random numbers generated by linear recurrence modulo two, Math. of Comp., 19 (1965) 201–209.
6. E. J. Watson, Primitive polynomials (mod 2), Math. of Comp., 16 (1962) 368–369.

A CLASS OF FUNCTIONS HAVING ELEMENTARY INTEGRALS FOR ARC LENGTH

W. G. DOTSON, JR. and R. G. SAVAGE, North Carolina State University

Arc length is frequently introduced as an application of integration before one studies special techniques of integration (e.g., integration by parts, trigonometric substitutions, etc.), and when this is the case, the problem section on arc length is usually rather meager. The purpose of this note is to indicate how one can obtain a rather large class of functions having quite elementary integrals for arc length. It is of interest to note that most of the standard examples, e.g., cosh x, ln(sec x), are obtainable as special cases of our method. In fact, we shall show that all functions (with nonnegative, continuous, "admissible" derivatives) having "elementary" integrals for arc length are obtainable by our method.

If $f(x)$ is a continuously differentiable real function on $[a, b]$, then $L[f](x)$, the arc length integral function for $f(x)$, is given by $L[f](x)=\int_a^x\{1+[f'(t)]^2\}^{1/2}dt$, $a\leqq x\leqq b$. Thus, L is a nonlinear self-mapping of the space of continuously differentiable functions on $[a, b]$. A function $\phi(x)$ is said to be *admissible* provided the antiderivative $D^{-1}[\phi(x)]$ can be found by basic integration formulas (i.e., without resorting to special techniques as indicated above). Integrals of the form $\int_a^x\phi(t)dt$ are said to be *elementary* provided the integrand $\phi(x)$ is admissible.

THEOREM 1. *If $\phi(x)$ and $\phi(x)^{-1}=1/\phi(x)$ are positive and continuous on $[a, b]$ and admissible, and if $f(x)$ is defined by the elementary integral* the irreducible polynomial x^2+x+1 over $\boldsymbol{Z}_2$. With such a list available, we may consider the quotient fields modulo the ideal generated by a prime in some detail. In general the quotient field will have a cyclic multiplication group of nonzero elements (see [3] p. 317) and the additive group will be the product of as many copies of $\boldsymbol{Z}_2$ as the degree of the prime. Thus we may obtain the Galois fields $GF[2^n]$ in this manner.

Since this domain is of characteristic two ($\mathbf{x}\oplus\mathbf{x}=\mathbf{0}$ for all $\mathbf{x}$), it follows that the equation $\mathbf{x}^n\oplus\boldsymbol{y}^n=\boldsymbol{z}^n$ has solutions in positive integers (under digital arithmetic) whenever n is a power of two; furthermore, given any pair $\mathbf{x}, \boldsymbol{y}$ the solution is uniquely given by $\boldsymbol{z}=\mathbf{x}\oplus\boldsymbol{y}$ (see [2] Problem 12, p. 438). While Fermat's Last Theorem does not hold in this domain, it is interesting to ask if perhaps there are no solutions when n is not a power of two. We note that $\mathbf{x}^3\oplus\boldsymbol{y}^3=(\mathbf{x}\oplus\boldsymbol{y})\times(\mathbf{x}^2\oplus\mathbf{x}\boldsymbol{y}\oplus\boldsymbol{y}^2)$, but no solutions have been found.

The preparation of this paper was supported in part by the National Science Foundation under GP-4432 to the University of Wisconsin.

References

1. C. Berge, The Theory of Graphs, Wiley, New York, 1962.
2. W. E. Deskins, Abstract Algebra, Macmillan, New York, 1964.
3. I. N. Herstein, Topics in Algebra, Blaisdell, New York, 1964.
4. D. J. Lewis, Introduction to Algebra, Harper and Row, New York, 1965.
5. R. C. Tausworthe, Random numbers generated by linear recurrence modulo two, Math. of Comp., 19 (1965) 201–209.
6. E. J. Watson, Primitive polynomials (mod 2), Math. of Comp., 16 (1962) 368–369.

A CLASS OF FUNCTIONS HAVING ELEMENTARY INTEGRALS FOR ARC LENGTH

W. G. DOTSON, JR. and R. G. SAVAGE, North Carolina State University

Arc length is frequently introduced as an application of integration before one studies special techniques of integration (e.g., integration by parts, trigonometric substitutions, etc.), and when this is the case, the problem section on arc length is usually rather meager. The purpose of this note is to indicate how one can obtain a rather large class of functions having quite elementary integrals for arc length. It is of interest to note that most of the standard examples, e.g., cosh x, ln(sec x), are obtainable as special cases of our method. In fact, we shall show that all functions (with nonnegative, continuous, "admissible" derivatives) having "elementary" integrals for arc length are obtainable by our method.

If $f(x)$ is a continuously differentiable real function on $[a, b]$, then $L[f](x)$, the arc length integral function for $f(x)$, is given by $L[f](x)=\int_a^x\{1+[f'(t)]^2\}^{1/2}dt$, $a\leqq x\leqq b$. Thus, L is a nonlinear self-mapping of the space of continuously differentiable functions on $[a, b]$. A function $\phi(x)$ is said to be *admissible* provided the antiderivative $D^{-1}[\phi(x)]$ can be found by basic integration formulas (i.e., without resorting to special techniques as indicated above). Integrals of the form $\int_a^x\phi(t)dt$ are said to be *elementary* provided the integrand $\phi(x)$ is admissible.

THEOREM 1. *If $\phi(x)$ and $\phi(x)^{-1}=1/\phi(x)$ are positive and continuous on $[a, b]$ and admissible, and if $f(x)$ is defined by the elementary integral*

$$f(x) = C + \int_a^x [A\phi(t) - B\phi(t)^{-1}]dt, \qquad a \leqq x \leqq b,$$

where $A>0$, $B>0$, *and* $4AB=1$, *then* $L[f](x)$ *is given by the elementary integral*

$$L[f](x) = \int_a^x [A\phi(t) + B\phi(t)^{-1}]dt, \qquad a \leqq x \leqq b.$$

Proof. We have $f'(x) = A\phi(x) - B\phi(x)^{-1}$, so that

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + [A^2\phi(x)^2 - 2AB + B^2\phi(x)^{-2}] = A^2\phi(x)^2 + (1/2) + B^2\phi(x)^{-2} \\ &= [A\phi(x) + B\phi(x)^{-1}]^2. \end{aligned}$$

Hence

$$\{1 + [f'(x)]^2\}^{1/2} = \left| A\phi(x) + B\phi(x)^{-1} \right| = A\phi(x) + B\phi(x)^{-1}.$$

Many examples are now easily constructed. With $\phi(x)=x$, $x>0$, we get $f(x)=Ax^2/2-B\ln(x)$, $4AB=1$; and with $\phi(x)=x^p(p^2\neq 1,\ x>0)$ we get $f(x) = Ax^{p+1}/(p+1)+B/(p-1)x^{p-1}$, $4AB=1$. Some specific numerical examples of these would be: $f(x)=x^2/8-\ln(x)$, $f(x)=x^3/6+1/2x$, $f(x)=x^4/4+1/8x^2$. With $\phi(x)=c^x(c>0,\ c\neq 1)$, we get $f(x)=(Ac^x+Bc^{-x})/\ln c$, $4AB=1$. Of course, a special case of this is $f(x)=\frac{1}{2}(e^x+e^{-x})=\cosh x$. With $\phi(x)=\tan x$, $0<x<\pi/2$, we get $f(x)=A\ln(\sec x)+B\ln(\csc x)$, $4AB=1$; with $\phi(x)=\cos x$, $0\leqq x<\pi/2$, we get $f(x)=A\sin x-B\ln(\sec x+\tan x)$, $4AB=1$; with $\phi(x)=\sec x+\tan x$, $-\pi/2<x<\pi/2$, $(\phi(x)^{-1}=\sec x-\tan x)$ and $A=B=1/2$, we get $f(x)=\ln(\sec x)$.

THEOREM 2. *Suppose* $f'(x)$ *is nonnegative, continuous, and admissible, and suppose* $\int_a^x\{1+[f'(t)]^2\}^{1/2}dt$ *is an elementary integral. Then there exists a positive, continuous, admissible function* $\phi(x)$ *such that* $\phi(x)^{-1}$ *is also positive, continuous, and admissible, and such that* $f'(x)=(1/2)\phi(x)-(1/2)\phi(x)^{-1}$; *and hence* $f(x)$ *and* $L[f](x)$ *are given by the elementary integrals*

$$f(x) = f(a) + \int_a^x [(1/2)\phi(t) - (1/2)\phi(t)^{-1}]dt, \qquad a \leqq x \leqq b,$$

and

$$L[f](x) = \int_a^x [(1/2)\phi(t) + (1/2)\phi(t)^{-1}]dt, \qquad a \leqq x \leqq b.$$

Proof. Letting $g(x)=[1+f'(x)^2]^{1/2}$, we have that $g(x)$ is positive, continuous, and admissible. Define $\phi(x)$ by $\phi(x)=g(x)+[g(x)^2-1]^{1/2}=g(x)+f'(x)$, since $f'(x)\geqq 0$. Since $f'(x)$ is nonnegative, continuous, and admissible, it is clear that $\phi(x)$ is positive, continuous, and admissible. One checks that $(1/2)\phi(x) -(1/2)\phi(x)^{-1}=(1/2)[\phi(x)^2-1]/\phi(x)=f'(x)$. Finally, $\phi(x)^{-1}$ is positive and continuous (since $\phi(x)$ is positive and continuous); and since $\phi(x)^{-1}=\phi(x)-2f'(x)$, and $\phi(x)$ and $f'(x)$ are admissible, we have that $\phi(x)^{-1}$ is admissible. Hence $f(x)$ is given by the first elementary integral above, and by Theorem 1 we then get that $L[f](x)$ is given by the second elementary integral above.

A NOTE ON ARC LENGTH

JOHN T. WHITE, Texas Technological College

Upon reading an article by Kaucher [1] one is led to consider motivating arc length of a curve described by the graph of f on $[a, b]$ by approximating the curve by line segments which are tangent to the curve and adding these up.

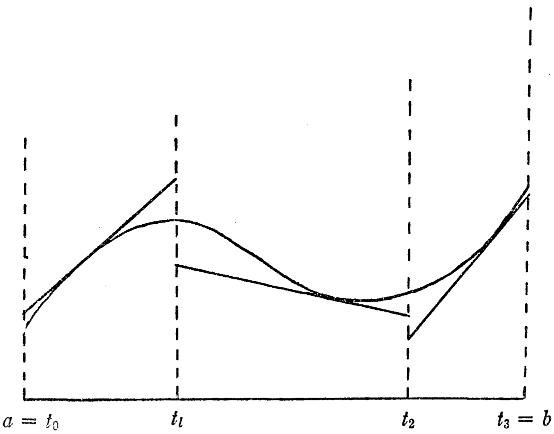


FIG. 1.

More precisely we have

DEFINITION. *Let f' be continuous on $[a, b]$; then the curve described by the graph on f on $[a, b]$ has upper length L if L is the greatest lower bound of*

$$\sum_{i=0}^{n-1} \sqrt{[f'(\eta_i)\Delta t_i]^2 + [\Delta t_i]^2} \tag{1}$$

taken over all partitions $P = \{t_0, t_1, \cdots, t_n\}$ of $[a, b]$, where η_i is a point in $[t_i, t_{i+1}]$ where $|f'|$ assumes its maximum value on $[t_i, t_{i+1}]$.

Thus, each term in (1) is the length of the tangent line segment between t_i and t_{i+1} having maximum absolute slope.

One then defines the lower length l in an analogous fashion and defines the curve to have length if both l and L exist and are equal, in which case the length is taken to be this common value.

THEOREM. *If f' is continuous on $[a, b]$, then the curve described by the graph of f has length.*

Proof. It is seen immediately from integration theory that L is the upper integral of $\sqrt{[f'(t)]^2+1}$ over $[a, b]$ and l is the lower integral. Hence, the length exists and is

$$\int_a^b \sqrt{[f'(t)]^2 + 1}\, dt.$$

Reference

1. John Kaucher, A theorem on arc length, this MAGAZINE, 42 (1969) 132–133.

Elementary Mathematics. By Donald D. Paige, Robert E. Willcutt, and Jerry M. Wagenblast. Prindle, Weber and Schmidt, Boston, 1969. xii+264 pp. $8.50. *Changing Bases, Mathematical Systems, Probability, Factors and Primes, Sentence Solution, Some Basic Statistics. Six Programmed Supplements.* By Donald D. Paige and Ian D. Beattie, Prindle, Weber and Schmidt, Boston, 1969. Approximately 32 pages each. 95¢ ea. (paper), $3.00 for the set.

This book is an attempt to study numbers and the number systems of arithmetic in a fashion which can be understood and followed by prospective elementary school teachers whose background in high school mathematics is very limited. Various numeration systems are discussed and then a brief study of sets is made with some applications to elementary probability. A chapter is devoted to each of the systems: whole numbers, integers, rationals, and reals.

It seemed to this reviewer that the explanations of the algorithms for the operations on the whole numbers were not clear enough to avoid confusion among many students. The same is true for the algorithm for extracting the square root of a number. The relationship of the real number system to the rational number system is not made clear as it easily could have been by a more careful explanation of the one-one correspondence possible between the real numbers and the points of the number line.

The attractive feature of the book is the strength of the sets of problems provided for the student. These do serve to get him involved in the discussion and not only pave the way for further work in this book but also should whet his appetite for more study and investigation. The last chapter of the book is programmed and there are also available six supplementary pamphlets in programmed form which may be used either with this book or with any other at this level. The programming is well done. Particularly successful is the pamphlet on probability.

As a text for the first course in mathematics for future teachers of elementary school it has possibilities. As a book for self-study it has serious limitations as noted above.

R. S. Pieters, Phillips Academy

The Teaching of Mathematics. Essays by A. Ya. Khinchin. Edited by B. V. Gnedenko. Translated from the Russian by W. Cochrane and D. Vere-Jones. American Elsevier, N. Y., 1968. xx+167 pp. $9.50.

The main contents of this book are four articles written by A. Ya. Khinchin for Russian mathematics teachers in the period 1938–1949. Sophisticated American readers will enjoy reading them and especially noting what a leading mathematician had to say about reform in the teaching of school mathematics before the first large scale organized curriculum program in mathematics teaching in the United States was initiated (1951). Readers of the Mathematics Teacher, the American Mathematical Monthly, and the yearbooks of the National Council of Teachers of Mathematics in the period 1938–1950 will find many familiar ideas. But there is a significant difference. These ideas were

Elementary Mathematics. By Donald D. Paige, Robert E. Willcutt, and Jerry M. Wagenblast. Prindle, Weber and Schmidt, Boston, 1969. xii+264 pp. $8.50. *Changing Bases, Mathematical Systems, Probability, Factors and Primes, Sentence Solution, Some Basic Statistics. Six Programmed Supplements.* By Donald D. Paige and Ian D. Beattie, Prindle, Weber and Schmidt, Boston, 1969. Approximately 32 pages each. 95¢ ea. (paper), $3.00 for the set.

This book is an attempt to study numbers and the number systems of arithmetic in a fashion which can be understood and followed by prospective elementary school teachers whose background in high school mathematics is very limited. Various numeration systems are discussed and then a brief study of sets is made with some applications to elementary probability. A chapter is devoted to each of the systems: whole numbers, integers, rationals, and reals.

It seemed to this reviewer that the explanations of the algorithms for the operations on the whole numbers were not clear enough to avoid confusion among many students. The same is true for the algorithm for extracting the square root of a number. The relationship of the real number system to the rational number system is not made clear as it easily could have been by a more careful explanation of the one-one correspondence possible between the real numbers and the points of the number line.

The attractive feature of the book is the strength of the sets of problems provided for the student. These do serve to get him involved in the discussion and not only pave the way for further work in this book but also should whet his appetite for more study and investigation. The last chapter of the book is programmed and there are also available six supplementary pamphlets in programmed form which may be used either with this book or with any other at this level. The programming is well done. Particularly successful is the pamphlet on probability.

As a text for the first course in mathematics for future teachers of elementary school it has possibilities. As a book for self-study it has serious limitations as noted above.

R. S. PIETERS, Phillips Academy

The Teaching of Mathematics. Essays by A. Ya. Khinchin. Edited by B. V. Gnedenko. Translated from the Russian by W. Cochrane and D. Vere-Jones. American Elsevier, N. Y., 1968. xx+167 pp. $9.50.

The main contents of this book are four articles written by A. Ya. Khinchin for Russian mathematics teachers in the period 1938–1949. Sophisticated American readers will enjoy reading them and especially noting what a leading mathematician had to say about reform in the teaching of school mathematics before the first large scale organized curriculum program in mathematics teaching in the United States was initiated (1951). Readers of the Mathematics Teacher, the American Mathematical Monthly, and the yearbooks of the National Council of Teachers of Mathematics in the period 1938–1950 will find many familiar ideas. But there is a significant difference. These ideas were being expressed in the United States by teachers and educators rather than by leading mathematicians. Prospective teachers of mathematics and others not familiar with the American literature would profit greatly by reading the book.

In his first article on basic concepts, Professor Khinchin stresses the basic importance of teaching so as not to conflict with later learning. He favors simplification, but never falsification. He advocates precise language—"to think in foggy terms cannot be easier than to think in precise terms" (p. 2). He believes that the schools should not attempt to answer the question "What is a number?" He says that when students ask the question, teachers should "Tell him that the question he has posed is one of the most difficult questions in the philosophy of science, one to which we are still far from having a complete answer." Even for the students who are unhappy with this answer it is better to wait with the answer for another year than "to substitute for that answer a surrogate which vulgarizes the problem" (p. 4). He favors introducing irrational numbers very much as it is usually done in the United States and he rejects a construction at the school level based on Weierstrass, Cantor, or Dedekind. " . . . it should be admitted frankly that the theory of irrational numbers in the full sense cannot be given in the secondary school."

One of the best sections is the one on the concept of limit. The author traces the historical development of the limit concept to determine which approach is best suited for the schools. The epsilon-delta approach is rejected in favor of a word approach. Khinchin prefers to say "the difference $|r-a_n|$ becomes and remains arbitrarily small as n increases without limit." However, he favors consideration of how large N needs to be so that $|r-a_n|$ should be less than some small number.

The second article on Mathematical Definitions is excellent and should be read by high school teachers. The discussion of the difference between a definition and a description and when it is desirable to present each is particularly good. It is refreshing that a mathematician of Khinchin's standing had such understanding of teaching problems.

The weakest chapter is on the Educative Effects of Mathematics Lessons. While it has some very good material on teaching students to avoid false generalizations and ill-founded analogies, to learn the necessity for full disjunction, and to desire complete and consistent classification, it also has some material which modern psychology does not support. The author states that "honesty in thought, having become for the mathematician an inexorable law of his scientific thinking and professional (in particular teaching) activities, influences him in all aspects of his life—from abstract discussions to everyday behavior" (p. 93). He continues (p. 93), "I must admit that I myself am organically incapable of admitting some assertion (even of an everyday, practical nature) unless I am in possession of a proof admitting no objections," and then makes unproved (and probably unprovable) assertions about transfer: "Conscientious and serious work on building up and consolidating knowledge in any scientific field . . . inevitably develops in the pupil corresponding qualities of character . . . " (p. 94).

In this same article Khinchin is on much firmer ground when he discusses the relationship of mathematics teaching to the development of patriotism and as an instrument of propaganda. He states (p. 95), "It is necessary to say straight out that . . . mathematics cannot directly, through its own material or content, serve as an instrument of propaganda for any such concrete cause as the beauty or greatness of the homeland." However, he does discuss appropriately, so it seems to the reviewer, how pride in homeland can be developed by studying the history of Russian and Soviet mathematicians.

Valuable additions to this book are the two appendices, a biography (with bibliography) by B. V. Gnedenko and A. I. Markushevich, and a description of Russian schools and school mathematics by D. Vere-Jones.

R. E. PINGRY, University of Illinois

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, MURRAY S. KLAMKIN, Ford Scientific Laboratory, Dearborn, Michigan

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before June 1, 1970.

PROPOSALS

747. *Proposed by J. A. H. Hunter, Toronto, Canada.*

The usual conditions apply to this alphametic. We have a prime $G\ A\ M\ E$ here, so what is this $G\ A\ M\ E$?

$$\begin{array}{cccc} & & & A \\ M & A & T & H \\ M & A & G' & S \\ M & A & T & H \\ \hline G & A & M & E \end{array}$$

In this same article Khinchin is on much firmer ground when he discusses the relationship of mathematics teaching to the development of patriotism and as an instrument of propaganda. He states (p. 95), "It is necessary to say straight out that . . . mathematics cannot directly, through its own material or content, serve as an instrument of propaganda for any such concrete cause as the beauty or greatness of the homeland." However, he does discuss appropriately, so it seems to the reviewer, how pride in homeland can be developed by studying the history of Russian and Soviet mathematicians.

Valuable additions to this book are the two appendices, a biography (with bibliography) by B. V. Gnedenko and A. I. Markushevich, and a description of Russian schools and school mathematics by D. Vere-Jones.

R. E. PINGRY, University of Illinois

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, MURRAY S. KLAMKIN, Ford Scientific Laboratory, Dearborn, Michigan

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before June 1, 1970.

PROPOSALS

747. *Proposed by J. A. H. Hunter, Toronto, Canada.*

The usual conditions apply to this alphametic. We have a prime $G\ A\ M\ E$ here, so what is this $G\ A\ M\ E$?

$$\begin{array}{cccc} & & & A \\ M & A & T & H \\ M & A & G' & S \\ M & A & T & H \\ \hline G & A & M & E \end{array}$$

748. *Proposed by Marlow Sholander, Case Western Reserve University.*

For $2x = b - a > 0$ it is well known that $2^a + 2^b > 2^{b+1-x}$. Show that $2^a + 2^b < 2^{b+p}$ where $p = 1/(1+2x^2)$.

749. *Proposed by Simeon Reich, Israel Institute of Technology, Haifa, Israel.*

Let ABC be an acute-angled triangle. Prove that $\frac{1}{2} < (h_a + h_b + h_c)/(a+b+c) < 1$, where h_a, h_b, h_c are the altitudes of the triangle and a, b, c its sides.

750. *Proposed by Charles W. Trigg, San Diego, California.*

Evaluate the determinant

$$\begin{vmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & a^2 - b^2 - c^2 + d^2 \end{vmatrix}$$

751.* *Proposed by Zalman Usiskin, University of Michigan.*

Prove or disprove: If a transformation of the plane maps each connected set onto a connected set, then the transformation is continuous.

752. *Proposed by Norman Schaumberger, Bronx Community College.*

Prove that

$$\lim_{n \to 0} \sum_{k=1}^{n-1} \left(\tan \frac{k\pi}{2n} / n \right)^p$$

converges for $p > 1$ and diverges for $p \leqq 1$.

753. *Proposed by Michael J. Martino, Temple University, and John DeJoice, Aries Corporation, McLean, Virginia.*

Define $f(n) = C(n) - C(n-1)$ for all $n \geqq 2$ where

$$C(n) = \sum_{i=1}^{[\sqrt{n}]} [(n/i) - (i-1)].$$

(The brackets indicate the greatest integer.)
Show that $f(n) = 1$ implies n is prime.

SOLUTIONS

Late Solutions

Philip Haverstick, Fort Belvoir, Virginia: **720**; *John R. Ventura, Jr., U.S. Naval Underwater Weapons Research and Engineering Station, Newport, Rhode Island:* **719** *and* **723**; *Thomas F. Parsons, Eastern Washington State College:* **719**.

Another Triangular Inequality

725. [May, 1969] *Proposed by W. J. Blundon, Memorial University of Newfoundland.*

Prove that for every triangle ABC,

$$(\sin A+\sin B+\sin C)/\sin A \sin B \sin C \geqq 4$$

with equality only for the equilateral triangle.

I. *Solution by L. Carlitz, Duke University.*

Let R denote the radius of the circumcircle. Then $a=2R \sin A$, etc., and the stated inequality becomes

$$(a+b+c)R^2 \geqq abc. \tag{1}$$

Since

$$a+b+c=2s, \qquad abc=4R\Delta,$$

where Δ is the area of ABC, (1) becomes

$$Rs \geqq 2\Delta = rs, \tag{2}$$

where r is the radius of the inscribed circle. But (2) is equivalent to the familiar inequality

$$R \geqq 2r.$$

II. *Solution by Dewey C. Duncan, Los Angeles, California.*

Since $A+B+C=\pi$ the given fraction readily reduces to

$$\frac{1}{2 \sin (A/2) \sin (B/2) \sin\{(A+B)/2\}} = 1/D$$

$$\frac{\partial D}{\partial A} = \sin (B/2) \cos(A+B/2) = 0$$

$$\frac{\partial D}{\partial B} = \sin (A/2) \cos(B/2+A) = 0$$

which yield the only acceptable solution $A=B=\pi/3$, producing an absolute maximum value for D, namely, 1/4. Accordingly, the given fraction has an absolute minimum value, namely, 4 for the values $A=B=C=\pi/3$.

III. *Solution by Leon Bankoff, Los Angeles, California.*

It is known that

$$\sum \frac{1}{h_a} = \frac{1}{r} \geqq \frac{2}{R}$$

with equality only for the equilateral triangle.

Hence

$$\sum \frac{2R}{h_a} \geqq 4$$

or

$$(\sin A + \sin B + \sin C)/\sin A \sin B \sin C \geqq 4.$$

Also solved by Andrew N. Aheart, West Virginia State College; Leon Bankoff, Los Angeles, California (five additional solutions); Arthur Bolder, Brooklyn College; John L. Brown, Jr., Pennsylvania State University; Donald Chand, Lockheed-Georgia Company; Mannis Charosh, Brooklyn, New York; Robert W. Chilcote, Bedford High School, Ohio; Mickey Dargitz, Ferris State College, Big Rapids, Michigan; Harry M. Gehman, SUNY at Buffalo, New York; Michael Goldberg, Washington, D.C.; Louise Grinstein, New York, New York; Yul Jay Inn, Aragon High School, San Mateo, California; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lew Kowarski, Morgan State College, Maryland; E. E. Morrison, Kings College, University of Aberdeen, Scotland; Cecil G. Phipps, Cookeville, Tennessee; Thomas L. Power, Eastern Washington State College; Simeon Reich, Israel Institute of Technology, Haifa, Israel; Henry J. Ricardo, Yeshiva University, New York; Harry Sitomer, C. W. Post College, New York; E. P. Starke, Plainfield, New Jersey; E. F. Schmeichel, College of Wooster, Ohio; P. D. Thomas, Naval Research Laboratory, Washington, D.C.; John R. Ventura, Jr., U.S. Naval Station, Newport, Rhode Island; K. Duane Wait, Proctor and Gamble Company, Cincinnati, Ohio; John W. Wrench, Jr., Naval Ship Research and Development Center, Washington, D.C.; Alexander Zujus, Chicago, Illinois; and the proposer.

Planetary Alphametic

726. [May, 1969] *Proposed by Willy Enggren, Copenhagen, Denmark.*

Solve the following cryptarithm:

$$\begin{array}{ccccccc} & & & & T & H & E \\ & & E & A & R & T & H \\ & & V & E & N & U & S \\ & S & A & T & U & R & N \\ & U & R & A & N & U & S \\ \hline N & E & P & T & U & N & E \end{array}$$

Solution by Kenneth M. Wilke, Topeka, Kansas.

In the following discussion, all congruences are taken to be for (mod 10). Clearly $N=1$ since $N=2$ or $N=0$ leads to later contradictions. Considering the columns from right to left, the first column requires $E+H+2S+1\equiv E$ or $H+2S\equiv 9$. This leads to $H=5$, $S=7$ since all other sets of values of (H, S) lead to contradictions.

The second column implies $2+5+2U+T+R\equiv 1$ and the third column implies $2+T+R+U+C_2\equiv V$ where C_2 is the carry-over from the second column.

Hence $2U\equiv 6+C_2$ so C_2 is even and $C_2=2$.

Hence $U=4$. ($U=9$ implies $C_2>2$)

Hence $T+R=6$ so that T, $R=6$, 0 in some order. The remaining unassigned digits are 2, 3, 8, 9. The fourth column requires $1+2A+E\equiv 0$ so $A=8$, $E=3$. The fifth column requires $2+E+V+A+R\equiv P$ or $3+V+R\equiv P$ where $R=0$ or 6 and P, $V=2$, 9 in some order. This is possible only if $P=2$, $V=9$, $R=0$; hence $T=6$. The solution unique is:

				6	5	3
		3	8	0	6	5
		9	3	1	4	7
	7	8	6	4	0	1
	4	0	8	1	4	7
1	3	2	6	4	1	3

Also solved by Richard L. Breisch, University of Colorado; Charles Chouteau, West Virginia State College; Frank DeStefano, Jamaica, New York; Dewey C. Duncan, Los Angeles, California; Alan H. Feireson and William A. Schmidt, Texas A and M University (jointly); Harry M. Gehman, SUNY at Buffalo, New York; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; Philip Haverstick, Fort Belvoir, Virginia; J. A. H. Hunter, Toronto, Canada; Yul Jay Inn, Aragon High School, San Mateo, California; David Isaacs, Valley High School, Valley Station, Kentucky; Alfred Kohler, Long Island University; R. Landeaux, New York University; Peter A. Lindstrom, Genesee Community College, New York; John W. Milsom, Butler County Community College, Pennsylvania; S. Ron Oliver, Morningside College, Iowa; Thomas L. Parsons, Eastern Washington State College; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; Bernard J. Portz, Jesuit College, St. Bonifacius, Minnesota; Simeon Reich, Israel Institute of Technology, Haifa, Israel; Norman A. Robins, Flossmoor, Illinois; E. F. Schmeichel, College of Wooster, Ohio; Charles W. Trigg, San Diego, California; John R. Tucker, Washington College, Maryland; John R. Ventura, Jr., U.S. Naval Station, Newport, Rhode Island; Kenneth L. Yocom, South Dakota State University; and the proposer.

A Log Related Expression

727. [May, 1969] *Proposed by John E. Prussing, University of California, San Diego.*

a] What is the range of positive values of x, such that for a given x, the only positive value of y which satisfies the equation $y^x = x^y$ is the trivial solution $y = x$?

b] For those positive values of x for which nontrivial positive solutions for y exist, how many solutions are there?

c] If a value of x is selected at random from the open interval $(0, e)$, what is the probability that a nontrivial solution for y lies in the same interval?

Solution by Arnold O. Allen, IBM Corporation, Los Angeles, California.

Since $x^y = \exp(y \ln x)$ for all real x and y, where exp is the exponential function, to compare x^y and y^x we need only compare $y \ln x$ and $x \ln y$. The exponential function is a strictly increasing function so that $x^y < y^x$, if and only if, $\ln x/x < \ln y/y$; $x^y = y^x$, if and only if, $\ln x/x = \ln y/y$; and $x^y > y^x$, if and only if, $\ln x/x > \ln y/y$. Thus to answer (a), (b) and (c) of this problem we need to consider the function g defined for all positive x by $g(x) = \ln x/x$. Now $g'(x) = (1 - \ln x)/x^2$ and $g''(x) = (-3 + 2 \ln x)/x^3$.

Therefore, g is negative for $0 < x < 1$, is strictly increasing for $0 < x \leqq e$, attains a maximum value of $1/e$ at e, and is strictly decreasing for $x \geqq e$. In addition, $g(1) = 0$ and $g(x)$ approaches the x-axis asymptotically as x becomes arbitrarily large. Therefore, the answer to (a) is that the equation $y^x = x^y$ has only the trivial solution $y = x$ only when $0 < x \leqq 1$ or when $x = e$. The answer to (b) is that when $1<x$, but $x\neq e$, there exists exactly one nontrivial solution of $y^x=x^y$. Let y be this solution. Then, if $1<x<e$, we have $y>e$; while, if $x>e$, then we have $1<y<e$. The answer to (c) is thus seen to be 0, since nontrivial solutions of $y^x=x^y$ exist in this case only if $1<x<e$; and then the corresponding y is larger than e.

Also solved by Richard L. Breisch, University of Colorado; Michael Goldberg, Washington, D.C.; Yul Jay Inn, Aragon High School, San Mateo, California; Arthur Marshall, Madison, Wisconsin; Steve Rohde, Lehigh University; E. F. Schmeichel, College of Wooster, Ohio; E. P. Starke, Plainfield, New Jersey; Kenneth L. Yocom, South Dakota State University; and the proposer.

Marshall found a reference to the problem in an article by E. J. Moulton, in the MONTHLY, 23 (1916) 233–237.

Conditional Convergence

728. [May, 1969] *Proposed by G. L. N. Rao, J. C. College, Jamshedpur, India.*

Find the sum of the infinite series:

$$\frac{x-2}{x^2-x+1}+\frac{2x^2-4}{x^4-x^2+1}+\frac{4x^4-8}{x^8-x^4+1}+\cdots$$

when $|x|>1$.

Solution by E. F. Schmeichel, College of Wooster, Ohio.

Let $f(x)$ denote the desired sum. Then

$$\begin{aligned} f(x) &= \frac{(x+1)(x-2)}{x^3+1}+\frac{2(x^2+1)(x^2-2)}{x^6+1}+\frac{4(x^4+1)(x^4-2)}{x^{12}+1}+\cdots \\ &= \frac{x^2-x+1-3}{x^3+1}+\frac{2(x^4-x^2+1-3)}{x^6+1}+\frac{4(x^8-x^4+1-3)}{x^{12}+1}+\cdots \\ &= \sum_{n=0}^{\infty}\frac{2^n}{x^{2^n}+1}-3\sum_{n=0}^{\infty}\frac{2^n}{(x^3)^{2^n}+1}\,. \end{aligned}$$

But

$$\sum_{n=0}^{\infty}\frac{2^n}{x^{2^n}+1}=\frac{1}{x-1},\quad \text{if } |x|>1,$$

(L. B. W. Jolley, *Summation of Series*, No. 91). So

$$f(x)=\frac{1}{x-1}-\frac{3}{x^3-1}=\frac{x+2}{x^2+x+1}\,.$$

Also solved by Wray G. Brady, Slippery Rock State College, Pennsylvania; L. Carlitz, Duke University; Dewey C. Duncan, Los Angeles, California; John E. Hafstrom, California State College at San Bernardino; Herbert R. Leifer, Pittsburgh, Pennsylvania; Peter A. Lindstrom, Genesee Community College, Batavia, New York; Steve Rohde, Lehigh University; E. P. Starke, Plainfield, New Jersey; John W. Wrench, U.S. Naval Ship Research and Development Center, Washington, D.C.; Kenneth L. Yocom, South Dakota State University; Alexander Zujus, Chicago, Illinois; and the proposer.

A Trigonometric Inequality

729. [May, 1969] *Proposed by T. J. Burke, R.C.A., Moorestown, New Jersey.*

Prove that for any set of real numbers $\{T_i\}$, $(i=1, 2, \cdots, n)$,

$$\sum_{k,j=1}^{n} \cos(T_k - T_j) \geqq 0.$$

I. *Solution by Douglas Lind, Cambridge University, England.*

Let $z=e^{iT_1}+e^{iT_2}+\cdots+e^{iT_n}$. Then we have

$$\begin{aligned} 0 \leqq z\bar{z} &= \left(\sum_{k=1}^{n} e^{iT_k}\right)\left(\sum_{j=1}^{n} e^{-iT_j}\right) \\ &= \sum_{k,j=1}^{n} e^{i(T_k-T_j)} \\ &= \sum_{k,j=1}^{n} \cos(T_k - T_j). \end{aligned}$$

II. *Solution by J. Ernest Wilkins, Jr., Gulf General Atomic, Inc., San Diego, California.*

More generally, it is true that for any set of real numbers T_i, x_i

$$\begin{aligned} \sum_{k,j=1}^{n} x_k \cos(T_k - T_j)x_j &= \sum_{k,j=1}^{n} x_k\{\cos T_k \cos T_j + \sin T_k \sin T_j\}x_j \\ &= \left\{\sum_{k=1}^{n} x_k \cos T_k\right\}^2 + \left\{\sum_{k=1}^{n} x_k \sin T_k\right\}^2 \geqq 0. \end{aligned}$$

Also solved by Arthur R. Bolder, Brooklyn College; J. L. Brown, Jr., Pennsylvania State University; L. Carlitz, Duke University; Donald Chand, Lockheed-Georgia Company; William F. Fox, Moberly, Missouri; Michael Goldberg, Washington, D.C.; John E. Hafstrom, California State College at San Bernardino; Carl Hammer, UNIVAC, Washington, D.C.; Yul Jay Inn, Aragon High School, San Mateo, California; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; E. F. Schmeichel, College of Wooster, Ohio; Alexander Zujus, Chicago, Illinois; and the proposer.

Several solvers found essentially the same inequality in *Elementary Inequalities*, by D. S. Mitrinović, Groningen, 1964, p. 102.

Similar Rhombuses

730. [May, 1969] *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, New York.*

Prove that all rhombuses inscribed in a given rectangle are similar.

I. *Solution by Alexander Zujus, Chicago, Illinois.*

We extend the proof to show more: All rhombuses "inscribed" in two pairs of mutually perpendicular parallel lines are similar.

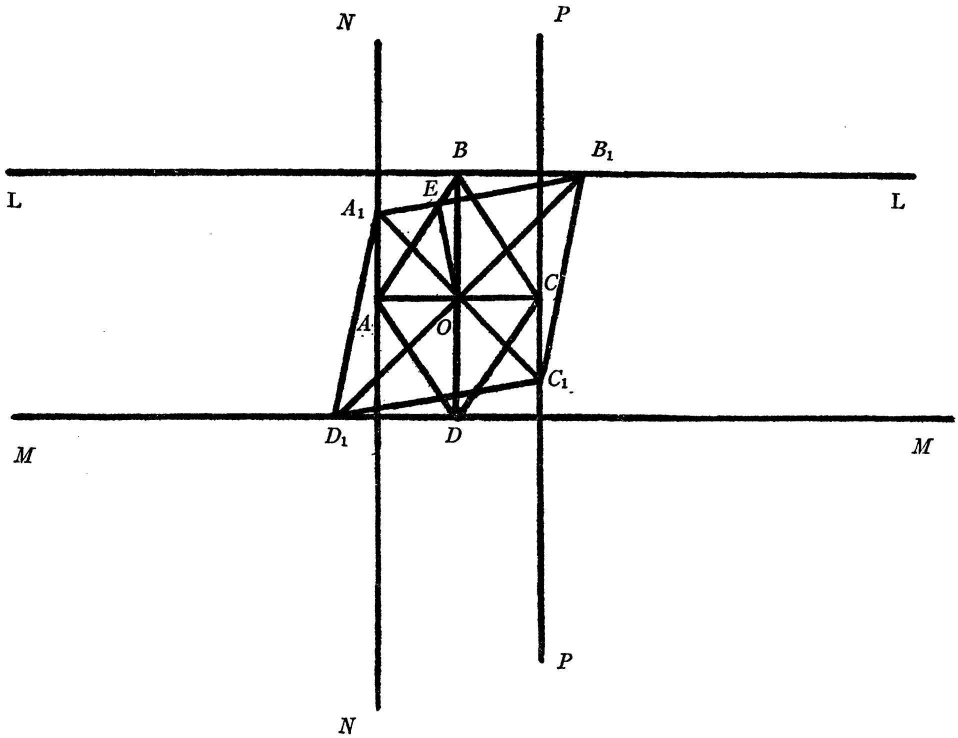


FIG. 1.

First (see figure), diagonals of all such rhombuses are concurrent at O, the center of the rectangle formed by the pairs of parallels. Let $ABCD$ be the rhombus whose vertices are at midpoints of the sides of this rectangle, and let $A_1B_1C_1D_1$ be any "inscribed" rhombus. Since $\Delta OAA_1 \sim \Delta OBB_1$ we have $(OA/OB) = (OA_1/OB_1)$ and it follows that $\Delta OAB \sim \Delta OAB_1$. That is, every "inscribed" is similar to the rhombus $ABCD$.

Some interesting consequences follow. Since $\measuredangle ABO = \measuredangle A_1B_1O$ and $\measuredangle OAB = \measuredangle OA_1B_1$, circles can be circumscribed about the quadrangles $OEBB_1$ and OAA_1E. Now, since $A_1A \perp AO$, it follows that $OE \perp A_1B_1$.

II. *Solution by William F. Fox, Moberly, Missouri.*

Consider a rhombus $ABCD$ in the plane with vertex A on the line $y = -a$, vertex B on the line $x = b$, vertex C on the line $y = a$, and vertex D on the line $x = -b$. (Any rhombus inscribed in a rectangle can be considered in this way, with one pair of sides of the rectangle the line $x = \pm b$, the other pair the lines $y = \pm a$.)

Clearly AC and BD intersect in the origin. (Note that x-axis bisects AC and y-axis bisects BD.)

Suppose C has coordinates (x_1, a), then A has coordinates $(-x_1, -a)$ and so AC has the equation $y = (a/x_1)x$. Hence, the equation of BD is $y = (-x_1/a)x$.

Thus the coordinates of B are $(b, -bx_1/a)$.

Hence, $OC=\sqrt{x_1^2+a^2}$, $OB=\sqrt{b^2+x_1^2b^2/a^2}$ and so tan $\angle CBO=b/a$. It follows that all rhombuses so constructed are similar.

Also solved by Leon Bankoff, Los Angeles, California; J. C. Binz, Bern, Switzerland; Santo M. Diano, Philadelphia, Pennsylvania; Dewey C. Duncan, Los Angeles, California (three solutions); Michael Goldberg, Washington, D.C.; Ned Harrell, Menlo-Atherton High School, California; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lew Kowarski, Morgan State College, Maryland; Herbert R. Leifer, Pittsburgh, Pennsylvania; Bernard J. Portz, Jesuit College, St. Bonifacius, Minnesota; Simeon Reich, Israel Institute of Technology, Haifa, Israel; E. F. Schmeichel, College of Wooster, Ohio; Harry Sitomer, C. W. Post College, New York; E. P. Starke, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; Paul J. Zwier, Grand Rapids, Michigan; and the proposer (two solutions).

A Factorial Polynomial

731. [May, 1969] *Proposed by Santosh Kumar, Ministry of Defense, New Delhi, India.*

a] Express a factorial polynomial of degree n, the coefficients of which are taken as the coefficients in the expansion of $(1+x)^n$, as an ordinary power polynomial and prove that the sum of the coefficients of this ordinary power polynomial is $(n+1)$.

b] Assuming binomial coefficients of a factorial polynomial of degree n, prove that the factorial polynomial behaves simply as the factorial function of the same degree with respect to the difference operator Δ.

Solution by Dewey C. Duncan, Los Angeles, California.

The factorial polynomial of degree n, $P_n(x)$ appears as

$$\text{(a)}\qquad 1+\sum_{i=1}^{i=n}(x)(x-1)(x-2)\cdots(x-n+i)\binom{n}{i-1}.$$

The sum of the coefficients of $P_n(x)$ must be $P_n(1)$ or

$$1+(1+n-n)\binom{n}{n-1}=1+n.$$

$$\text{(b)}\quad P_n(x+1)=1+\sum_{i=1}^{i=n}(x+1)(x)(x-1)(x-2)\cdots(x+1-n+i)\binom{n}{i-1}.$$

Now

$$\begin{aligned}\Delta P_n(x) &= P_n(x+1)-P_n(x)\\ &= n+\sum_{i=1}^{i=n-1}x(x-1)(x-2)\cdots(x-1-n+i)\binom{n}{i-1},\end{aligned}$$

i.e., $\Delta P_n(x)=n\cdot P_{n-1}(x)$, the difference function of $P_n(x)$.

$F(n)=n!=n\cdot(n-1)!=n\cdot F(n-1)$, the factorial function $F(n)$. Thus, the factorial polynomial of degree n, with respect to the difference operator behaves simply as the factorial function of degree n.

Also solved by James C. Hickman, University of Iowa; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan (partially); Simeon Reich, Israel Institute of Technology, Haifa, Israel; and the proposer. One unsigned solution was received.

A Condition for Rationality

732. [May, 1969] *Proposed by Douglas Lind, University of Virginia.*

Form the decimal number $x=x_0x_1x_2x_3 \cdots$ as follows. Let $x_0=1$ and x_n be the least positive remainder upon division by 9 of $x_0+x_1+ \cdots +x_{n-1}$. Show that x is rational.

Solution by Kenneth L. Yocom, South Dakota State University.

Since $x_{k+1}\equiv x_k+x_{k-1}+ \cdots +x_1+x_0\equiv x_k+x_k\equiv 2x_k \pmod 9$, the sequence $\{x_k\}$ is simply a geometric progression in Z_9 with common ratio 2.

Generalize by considering the sequence $\{x_{k+1}=r^kx_0\}$ with x_0, $r\in Z_n=\{0, 1, \cdots, n-1\}$ used to define the decimal $x_0\cdot x_1x_2x_3x_4 \cdots$ to the base $b>n$. Then by the finiteness of Z_n there exist x_k, x_{k+p} which are equal (after reduction modulo n). Hence $x_{k+j}\equiv r^jx_k\equiv r^jx_{k+p}\equiv x_{k+p+j} \pmod n$ for $j=0, 1, \cdots, p-1$. Hence the decimal has a repeating block of p digits and is rational.

Also solved by Arthur R. Bolder, Brooklyn College; Richard L. Breisch, University of Colorado; Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania; Dewey C. Duncan, Los Angeles, California; William F. Fox, Moberly, Missouri; Michael Goldberg, Washington, D.C.; John E. Hafstrom, California State College at San Bernardino; Philip Haverstick, Fort Belvoir, Virginia; J. E. Homer, Jr., Union Carbide Corp., Chicago, Illinois; J. A. H. Hunter, Toronto, Canada; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lew Kowarski, Morgan State College, Maryland; Herbert R. Leifer, Pittsburgh, Pennsylvania; Henry S. Lieberman, Newtonville, Massachusetts; Steve Rohde, Lehigh University; E. F. Schmeichel, College of Wooster, Ohio; Harry Sitomer, C. W. Post College, New York; E. P. Starke, Plainfield, New Jersey; J. Ernest Wilkins, Jr., Gulf General Atomic, Inc., San Diego, California; Alexander Zujus, Chicago, Illinios; and the proposer. One unsigned solution was received.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q468. If a, b, and c are positive, prove that $(a+b+c)^3\geqq 27abc$.

[*Submitted by Stanley Rabinowitz*]

Q469. Find a number such that its fractional part, its integral part, and the number itself are in geometric progression.

[*Submitted by David L. Silverman*]

Q470. Show that $2^{5n+1}+5^{n+2}$ is divisible by 27 for $n=0, 1, 2, \cdots$

[*Submitted by Alan Sutcliffe*]

Q471. Prove that 3^k is not the sum of two positive integer squares.

[*Submitted by Erwin Just and Norman Schaumberger*]

(Answers on page 36.)

the preceding section equations (c) and (d) simply require that $P=AA^+$ and $Q=A^+A$ be orthogonal projections. One easily observes that if in Section 3 we choose U to be the orthogonal complement of N and W to be the orthogonal complement of R, then we obtain the Moore-Penrose generalized inverse of A.

References

1. J. S. Frame, Matrix functions and applications, IEEE Spectrum, (1964) 209–220.
2. E. H. Moore, General analysis, Part I, Mem. Amer. Philos. Soc., I (1935).
3. R. A. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc., 51 (1955) 406–413.
4. D. W. Robinson, On the generalized inverse of an arbitrary linear transformation, Amer. Math. Monthly, 69 (1962) 412–416.

ON THE NUMBER OF SUBSETS OF A FINITE SET

DAVID S. GREENSTEIN, Northeastern Illinois State College

In Freshman courses in "Modern Mathematics", it is easy to find students who have difficulty grasping the combinatorial reasoning usually employed to show that an n-element set has 2^n subsets. I have found the following way of showing why adding a new element to a set doubles the number of subsets. This approach plus the obvious fact that the empty set has exactly one subset makes it easy to find the number of subsets of an n-element set in n easy steps.

Let A be a finite set and let $b \notin A$. Then each subset S of A "gives birth" to the two subsets S and $S\cup\{b\}$ of $A\cup\{b\}$. Furthermore, each subset of $A \cup\{b\}$ has a unique "birth" from some subset of A. Thus adding b has doubled the number of subsets.

ANSWERS

A468. Taking the cube root of both sides and dividing by 3 gives an obvious proof since the average of three positive numbers is not smaller than their geometric mean.

A469. $(x-[x])x=[x]^2$ or $x^2-[x]x-[x]^2=0$ and $x=[x](1+\sqrt{5})/2$.

Then $[x]+1\geqq[x](1+\sqrt{5})/2$ and $[x]\leqq 2/(\sqrt{5}-1)<2$. Thus $[x]=0$ or 1 and $x=0$ or $(1+\sqrt{5})/2$.

A470. $2^{5n+1}+5^{n+2}=2(27+5)^n+25+5^n=2k27+(2+25)5^n$.

A471. Without loss of generality, it may be assumed that k is the least integer such that $x^2+y^2=3^k$. Then $x^2+y^2\equiv 0 \pmod 3$ from which it follows that $x\equiv y\equiv 0 \pmod 3$. Therefore $x=3m$ and $y=3n$ which yields $(3m)^2+(3n)^2=3^k$ or $m^2+n^2=3^{k-2}$, a contradiction of the minimal condition on k. The conclusion follows.

(Quickies on page 56.)

MATH/70

BASIC MATHEMATICS, Form A, Second Series

M. Wiles Keller, Purdue University, and James H. Zant, Oklahoma State University

This text-workbook covers arithmetic, algebra, trigonometry, and the slide rule, each of which is preceded by a diagnostic test. Completely rewritten, this series conforms to current trends in vocabulary and approach. The text retains from the former series the presentation of mathematical rules in steps, with each step accompanied by an example written in a parallel column. Twelve achievement tests are placed throughout the book, and a wealth of updated exercises and word problems are provided. Answers to all exercises and diagnostic tests are given in the back of the book; answers to the achievement tests are available in a separate booklet.

paper, about 275 pages, March 1970

REVIEW ARITHMETIC

M. Wiles Keller, Purdue University, and James H. Zant, Oklahoma State University

REVIEW ARITHMETIC is a separate edition of the arithmetic portion of BASIC MATHEMATICS.

paper, about 114 pages, February 1970

ALGEBRA AND TRIGONOMETRY

Marvin Marcus, University of California at Santa Barbara, and Henryk Minc, University of California at Santa Barbara

Designed for students with a limited mathematics background, this readable precalculus text interweaves remedial material with topics from the theory of elementary functions. Included are several sections devoted to a review of algebraic manipulations, an elementary introduction to sets theory and number systems, and a slow and intuitive development of trigonometry. The functions approach is stressed. More than 200 worked examples are discussed in the text, and the exercises are extensive — all graded to levels of difficulty. The synopsis of material and the true-false questions that precede the exercises at the end of each section help the student understand the definitions and theorems.

Answers to the true-false quizzes and solutions to many of the exercises are worked out in full detail and comprise over 100 pages at the end of the book. Solutions to the remaining exercises are available in a separate manual.

cloth, about 368 pages, February 1970

COLLEGE ALGEBRA

Marvin Marcus, University of California at Santa Barbara, and Henryk Minc, University of California at Santa Barbara

COLLEGE ALGEBRA retains all the material from ALGEBRA AND TRIGNOMETRY with the exception of the chapter on trigonometry.

cloth, about 280 pages, February 1970

HOUGHTON MIFFLIN
110 Tremont Street, Boston, Massachusetts 02107
Atlanta/Dallas/Geneva, Ill./New York/Palo Alto